

# Two-points problem for an evolutional first order equation in Banach space

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## Abstract

Two-point nonlocal problem for the first order differential evolution equation with an operator coefficient in a Banach space  $X$  is considered. An exponentially convergent algorithm is proposed and justified in assumption that the operator coefficient is strongly positive and some existence and uniqueness conditions are fulfilled. This algorithm leads to a system of linear equations that can be solved by fixed-point iteration. The algorithm provides exponential convergence in time that in combination with fast algorithms on spatial variables can be efficient treating such problems. The efficiency of the proposed algorithms is demonstrated by numerical examples.

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*Key Words:* First order differential evolution equations in Banach space, nonlocal problem, unbounded operator coefficient, operator exponential, exponentially convergent algorithms

## 1 Introduction

The  $m$ -point initial (nonlocal) problem for a differential equation with the nonlocal condition  $u(t_0) + g(t_1; \dots; t_p; u) = u_0$  and a given function  $g$  on a given point set  $P = \{0 = t_0 < t_1 < \dots < t_p\}$ , is one of the important topics in the study of differential equations. Interest in such problems originates mainly from some physical problems with a control of the solution at  $P$ . For example, when the function  $g(t_1; \dots; t_p; u)$  is linear, we will have the periodic problem  $u(t_0) = u(t_1)$ . Problems with nonlocal conditions arise in the theory of physics of plasma [19], nuclear physics [14], mathematical chemistry [15], waveguides [12] etc. Two-point problem is also useful for considering the finale value problem [22].

Differential equations with operator coefficients in some Hilbert or Banach space can be considered as meta-models for systems of partial or ordinary differential equations and are suitable for investigations using tools of the functional analysis (see e.g. [4, 13]). Nonlocal problems can also be considered within this framework [2, 3].

Discretization methods for differential equations in Banach and Hilbert spaces were intensively studied in the last decade (see e.g. [5, 7, 8, 9, 16, 17, 20, 21, 26, 27] and the references therein). Methods from [7, 8, 9, 16, 17, 21, 26, 27] possess an exponential convergence rate, i.e. the error estimate in an appropriate norm is of the type  $\mathcal{O}(e^{-N^\alpha})$ ,  $\alpha > 0$  with respect to a discretization parameter  $N \rightarrow \infty$ . For a given tolerance  $\varepsilon$  such discretization provides optimal or nearly optimal computational complexity [7].

In the present paper we consider the problem

$$\begin{aligned} \frac{du(t)}{dt} + A_1(t)u(t) &= f_1(t), \\ u(0) + \alpha u(1) &= \varphi, \end{aligned} \tag{1.1}$$

where  $A_1(t)$  is a densely defined closed (unbounded) operator with the domain  $D(A_1)$  independent of  $t$  in a Banach space  $X$ ,  $\varphi$  is given vector and  $f_1(t)$  is given vector-valued function,  $\alpha \in \mathbb{R}$ . We suppose that the

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operator  $A_1(t)$  is strongly positive; i.e. there exists a positive constant  $M_R$  independent of  $t$  such that on the rays and outside a sector  $\Sigma_\theta = \{z \in \mathbb{C} : 0 \leq \arg(z) \leq \theta, \theta \in (0, \pi/2)\}$  the following resolvent estimate holds:

$$\|(zI - A_1(t))^{-1}\| \leq \frac{M_R}{1 + |z|}. \quad (1.2)$$

This assumption implies that there exists a positive constant  $c_\kappa$  such that ( see [6], p.103)

$$\|A_1^\kappa(t)e^{-sA_1(t)}\| \leq c_\kappa s^{-\kappa}, \quad s > 0, \quad \kappa \geq 0. \quad (1.3)$$

Our further assumption is that there exists a real positive  $\omega$  such that

$$\|e^{-sA_1(t)}\| \leq e^{-\omega s} \quad \forall s, t \in [0, 1] \quad (1.4)$$

(see [18], Corollary 3.8, p.12, for corresponding assumptions on  $A_1(t)$ ). Let us also assume that the following conditions hold true

$$\|[A_1(t) - A_1(s)]A_1^{-\gamma}(t)\| \leq L_{1,\gamma}|t - s| \quad \forall t, s, 0 \leq \gamma < 1, \quad (1.5)$$

$$\|A_1^\gamma(t)A_1^{-\gamma}(s) - I\| \leq L_\gamma|t - s| \quad \forall t, s \in [0, 1]. \quad (1.6)$$

We suppose also that

$$f_1(t) \in C(0, 1; X). \quad (1.7)$$

The aim of this paper is to construct an exponentially convergent approximation to the problem (1.1) for a differential equation with two-points nonlocal condition in abstract setting. The paper is organized as follows. In Section 2 we discuss the existence and uniqueness of the solution as well as its representation through input data. A numerical algorithm is presented in section 3. The main result of this section is theorem 3.3 about the convergence rate of the proposed discretization. The next section 4 we represent some numerical example which confirm theoretical results from the previous sections.

## 2 Existence and uniqueness of the solution

It is well known, that for  $\alpha = 0$  the problem (1.1) has a unique solution under the assumptions (1.2)-(1.7) (se e.g. [18, 13]). This solution can be write down as follows:

$$u(t) = U(t, 0)u(0) + \int_0^t U(t, s)f_1(s)ds = U(t, 0)\varphi + \int_0^t U(t, s)f_1(s)ds, \quad (2.1)$$

where  $U(t, s)$  is an evolution operator that corresponds to (1.1) for  $\alpha = 0$ .

Let us study conditions when there is unique solution for the two-points problem (1.1). We have from (2.1)

$$u(1) = U(1, 0)u(0) + \int_0^1 U(1, s)f_1(s)ds.$$

Substituting this expression into the nonlocal condition we obtain

$$u(0) = [I + \alpha U(1, 0)]^{-1} \left[ \varphi - \alpha \int_0^1 U(1, s)f_1(s)ds \right],$$

and for  $u(t)$  we have

$$u(t) = U(t, 0) [I + \alpha U(1, 0)]^{-1} \left[ \varphi - \alpha \int_0^1 U(1, s)f_1(s)ds \right] + \int_0^t U(t, s)f_1(s)ds. \quad (2.2)$$

It is necessary to establish conditions on  $\alpha$  for the existence of  $u(t)$ . In fact, we have to explore when exists  $[I + \alpha U(1, 0)]^{-1}$ . So, we obtain using estimate for  $U(t, s)$  (see e.g. [18, 13]).

$$\|[I + \alpha U(1, 0)]^{-1}\| \leq [1 - |\alpha| \|U(1, 0)\|]^{-1} \leq [1 - |\alpha|M]^{-1} \leq C,$$

for small enough  $\alpha$  ( $\alpha < M^{-1}$ ).

### 3 Numerical algorithm

We use the approach developed in [10] and [25] to construct numerical method for solving problem (1.1). First of all we change variable in (1.1) by  $t \rightarrow \frac{1+t}{2}$  and for  $v(t) = u\left(\frac{1+t}{2}\right)$  we have

$$\begin{aligned} \frac{dv(t)}{dt} + A(t)v(t) &= f(t), \\ v(-1) + \alpha v(1) &= \varphi, \end{aligned} \quad (3.1)$$

where  $A(t) = \frac{1}{2}A_1\left(\frac{1+t}{2}\right)$ ,  $f(t) = \frac{1}{2}f_1\left(\frac{1+t}{2}\right)$ ,

We choose a mesh  $\omega_n = \{t_k, k = 0, \dots, n\}$  of  $n + 1$  various points on  $[-1, 1]$  that are Chebyshev-Gauss-Lobatto nodes  $t_k = \cos\left(\frac{n-k}{n}\pi\right)$ , and set  $\tau_k = t_k - t_{k-1}$ . Let

$$\begin{aligned} \bar{A}(t) &= A_k = A(t_k), t \in (t_{k-1}, t_k], \quad k = \overline{1, n}, \\ A_0 &= A(-1). \end{aligned} \quad (3.2)$$

Let us rewrite the problem (3.1) in the equivalent form

$$\begin{aligned} \frac{dv}{dt} + \bar{A}(t)v &= [\bar{A}(t) - A(t)]v(t) + f(t), \quad t \in (-1, 1) \\ v(-1) &= \varphi - \alpha v(1). \end{aligned} \quad (3.3)$$

Note, that now all operators on the left side of these equations are constant on each subinterval and piece-wise constant on the whole interval  $[-1, 1]$ .

On each subinterval we can write down the equivalent to (3.3) integral equation

$$\begin{aligned} v(t) &= e^{-A_k(t-t_{k-1})}v(t_{k-1}) + \int_{t_{k-1}}^t e^{-A_k(t-s)}[A_k - A(t)]v(s)ds + \int_{t_{k-1}}^t e^{-A_k(t-s)}f(s)ds, \\ t &\in [t_{k-1}, t_k], \quad k = \overline{2, n}, \end{aligned} \quad (3.4)$$

$$v(t) = e^{-A_1(t+1)}[\varphi - \alpha v(1)] + \int_{-1}^t e^{-A_1(t-s)}[A_1 - A(t)]v(s)ds + \int_{-1}^t e^{-A_1(t-s)}f(s)ds, \quad t \in [-1, t_1]. \quad (3.5)$$

Let

$$P_n(t; v) = P_n v = \sum_{j=0}^n v(t_j)L_{j,n}(t), \quad (3.6)$$

be the interpolation polynomial for  $v(t)$  on the mesh  $\omega_n$ ,  $x = (x_0, \dots, x_n)$ ,  $x_i \in X$  given vector and

$$P_n(t; y) = P_n x = \sum_{j=0}^n x_j L_{j,n}(t) \quad (3.7)$$

the polynomial that interpolates  $x$ , where

$$L_{j,n}(s) = \frac{T'_n(s)(1-s^2)}{\frac{d}{ds}[(1-s^2)T'_n(s)]_{s=s_j}(s-s_j)}, \quad j = 0, \dots, n$$

are the Lagrange fundamental polynomials. Substituting  $P_n(s; x)$  for  $v(s)$ ,  $x_k$  for  $v(t_k)$  and then setting  $t = t_k$  in (3.4) we arrive at the following system of linear equations with respect to the unknown  $x_k$ :

$$\begin{aligned} x_0 + \alpha x_n &= \varphi, \\ x_k &= e^{-A_k \tau_k} x_{k-1} + \sum_{j=0}^n \alpha_{kj} x_j + \phi_k, \quad k = \overline{1, n}, \end{aligned} \quad (3.8)$$

which represents our algorithm. Here we use the notations

$$\begin{aligned}\alpha_{kj} &= \int_{t_{k-1}}^{t_k} e^{-A_k(t_k-s)} [A_k - A(s)] L_{j,n}(s) ds, \\ \phi_k &= \int_{t_{k-1}}^{t_k} e^{-A_k(t_k-s)} f(s) ds, \quad k = \overline{1, n}, \quad j = \overline{0, n},\end{aligned}\tag{3.9}$$

and suppose that we have an algorithm to compute these coefficients.

For the error  $z = (z_1, \dots, z_n)$ , with  $z_k = v(t_k) - x_k$  we have the relations

$$\begin{aligned}z_0 + \alpha z_n &= 0, \\ z_k &= e^{-A_k \tau_k} z_{k-1} + \sum_{j=0}^n \alpha_{kj} z_j + \psi_k, \quad k = \overline{1, n},\end{aligned}\tag{3.10}$$

where

$$\psi_k = \int_{t_{k-1}}^{t_k} e^{-A_k(t_k-s)} [A_k - A(s)] [v(s) - P_n(s; v)] ds, \quad k = \overline{1, n},\tag{3.11}$$

In order to represent algorithm (3.8) in a block-matrix form we introduce the matrix

$$S = \begin{pmatrix} I & 0 & 0 & \cdot & \cdot & \cdot & 0 & \alpha \sigma_0 \\ -\sigma_1 & I & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & -\sigma_2 & I & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -\sigma_n & I \end{pmatrix},\tag{3.12}$$

where  $\sigma_0 = A_0^\gamma A_n^{-\gamma}$ ,  $\sigma_k = e^{-A_k \tau_k} A_k^\gamma A_{k-1}^{-\gamma}$ ,  $k = \overline{1, n}$ , the matrix  $B = \{\tilde{\alpha}_{k,j}\}_{k,j=0}^n$  with  $\tilde{\alpha}_{k,j} = A_k^\gamma \alpha_{k,j} A_j^{-\gamma}$ ,  $k = \overline{1, n}$ ,  $j = \overline{0, n}$ , and  $\tilde{\alpha}_{0,j} = 0$ ,  $j = \overline{0, n}$ , the vectors

$$\tilde{x} = \begin{pmatrix} A_0^\gamma x_0 \\ A_1^\gamma x_1 \\ \cdot \\ A_n^\gamma x_n \end{pmatrix}, \quad \phi = \begin{pmatrix} A_0^\gamma \varphi \\ A_1^\gamma \phi_1 \\ \cdot \\ A_n^\gamma \phi_n \end{pmatrix}, \quad \tilde{z} = \begin{pmatrix} A_0^\gamma z_0 \\ A_1^\gamma z_1 \\ \cdot \\ A_n^\gamma z_n \end{pmatrix}, \quad \psi = \begin{pmatrix} 0 \\ A_1^\gamma \psi_1 \\ \cdot \\ A_n^\gamma \psi_n \end{pmatrix}.\tag{3.13}$$

It is easy to check that for the (left) inverse

$$S^{-1} = \delta (R_1 - R_2),\tag{3.14}$$

where

$$\begin{aligned}\delta &= (I + \alpha \sigma_0 \sigma_1 \dots \sigma_n)^{-1}, \\ R_1 &= \begin{pmatrix} I & 0 & \dots & 0 & 0 \\ \sigma_1 & I & \dots & 0 & 0 \\ \sigma_2 \sigma_1 & \sigma_2 & \dots & 0 & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot \\ \sigma_n \dots \sigma_1 & \sigma_n \dots \sigma_2 & \dots & \sigma_n & I \end{pmatrix}, \\ R_2 &= \alpha \sigma_0 \begin{pmatrix} 0 & \sigma_n \dots \sigma_2 & \sigma_n \dots \sigma_3 & \dots & \sigma_n & I \\ 0 & 0 & \sigma_1 \sigma_n \dots \sigma_3 & \dots & \sigma_1 \sigma_n & \sigma_1 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & \sigma_{n-1} \dots \sigma_1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.\end{aligned}$$

**Remark 3.1** Using results of [11, 7, 8] one can get a parallel and sparse approximations with an exponential convergence rate of the operator exponentials contained in  $S^{-1}$  and as a consequence a parallel and sparse approximation of  $S^{-1}$ .

We multiply the equations in (3.8) and the equation in (3.10) by  $A_k^\gamma$ ,  $k = \overline{0, n}$  and obtain

$$\begin{aligned} A_0^\gamma x_0 + \alpha A_0^\gamma x_n &= A_0^\gamma \varphi, \\ A_k^\gamma x_k &= e^{-A_k \tau_k} A_k^\gamma x_{k-1} + \sum_{j=0}^n \tilde{\alpha}_{kj} A_j^\gamma x_j + A_k^\gamma \phi_k, \quad k = \overline{1, n}, \end{aligned} \quad (3.15)$$

$$\begin{aligned} A_0^\gamma z_0 + \alpha A_0^\gamma z_n &= 0, \\ A_k^\gamma z_k &= e^{-A_k \tau_k} A_k^\gamma z_{k-1} + \sum_{j=0}^n \tilde{\alpha}_{kj} A_j^\gamma z_j + A_k^\gamma \psi_k, \quad k = \overline{1, n}, \end{aligned} \quad (3.16)$$

Then the systems (3.15), (3.16) can be written down in the matrix form using notations (3.12), (3.13) as

$$\begin{aligned} S\tilde{x} &= B\tilde{x} + \phi, \\ S\tilde{z} &= B\tilde{z} + \psi. \end{aligned} \quad (3.17)$$

Next, for a vector  $v = (v_1, v_2, \dots, v_n)^T$  and a block operator matrix  $A = \{a_{ij}\}_{i,j=1}^n$  we introduce a vector norm

$$\| \|v\| \| \equiv \| \|v\| \|_1 = \max_{1 \leq k \leq n} \|v_k\|,$$

and the consistent matrix norm

$$\| \|A\| \| \equiv \| \|A\| \|_1 = \max_{1 \leq i \leq n} \sum_{j=1}^n \|a_{i,j}\|.$$

Due to (1.6) we have  $\| \|A_k^\gamma A_{k-1}^{-\gamma}\| \| = \| \|A_k^\gamma A_{k-1}^{-\gamma} - I + I\| \| \leq 1 + L_\gamma \tau_k$ ,  $\| \sigma_0 \| = \| \|A_0^\gamma A_n^{-\gamma}\| \| \leq 1 + L_\gamma T$ . In our case  $T = 2$ . So, we have the following, using these estimates

$$\begin{aligned} \| \sigma_k \| &= \| \|e^{-A_k \tau_k} A_k^\gamma A_{k-1}^{-\gamma}\| \| \leq e^{-\omega \tau_k} \| \|A_k^\gamma A_{k-1}^{-\gamma}\| \| \leq e^{-\omega \tau_k} (1 + L_\gamma \tau_k), \\ \| \delta \| &= \| \|(I + \alpha \sigma_0 \sigma_1 \dots \sigma_n)^{-1}\| \| \leq (1 - |\alpha| \| \sigma_0 \| \| \sigma_1 \| \| \sigma_2 \| \dots \| \sigma_n \|)^{-1} \\ &\leq (1 - |\alpha| (1 + 2L_\gamma) e^{-\omega \tau_1} (1 + L_\gamma \tau_1) e^{-\omega \tau_2} (1 + L_\gamma \tau_2) \dots e^{-\omega \tau_n} (1 + L_\gamma \tau_n))^{-1} \\ &\leq \left( 1 - |\alpha| (1 + 2L_\gamma) e^{-2\omega} \left( 1 + \frac{2L_\gamma}{n} \right)^n \right)^{-1} \leq (1 - |\alpha| (1 + 2L_\gamma) e^{-2\omega} e^{2L_\gamma})^{-1} \leq c, \end{aligned}$$

for  $\alpha$  small enough.

In order to estimate the norm of matrix  $S$  we have to estimate the norms of matrices  $R_1, R_2$ . In [10] it was proved that for matrix similar to  $R_1$  the estimate  $\| \|R_1\| \| \leq cn$  holds true. Let us estimate the norm of matrix  $R_2$ .

$$\begin{aligned} \| \|R_2\| \| &\leq (1 + 2c) (1 + e^{-\omega \tau} (1 + c\tau) + \dots + [e^{-\omega \tau} (1 + c\tau)]^{n-1}) \\ &\leq (1 + 2c) \left( 1 + (1 + c\tau) + \dots + (1 + c\tau)^{n-1} \leq \frac{(1 + c\tau)^n - 1}{c\tau} \right) \leq (1 + 2c) \frac{e^{2c}}{c\tau} \leq cn. \end{aligned}$$

Using these estimates we obtain that

$$\| \|S^{-1}\| \| \leq cn. \quad (3.18)$$

It was proved an estimate for the matrix  $B$  in [10]:

$$\| \|B\| \| \leq cn^{\gamma-2} \ln(n). \quad (3.19)$$

So we can formulate the following assertion

**Lemma 3.2** *Let the assumptions (1.2)-(1.6) are fulfilled then the estimates (3.18), (3.19) hold true.*

Using (3.17) we have

$$\begin{aligned}\tilde{x} &= [E - S^{-1}B]^{-1} S^{-1}\phi, \\ \tilde{z} &= [E - S^{-1}B]^{-1} S^{-1}\psi,\end{aligned}\tag{3.20}$$

where  $E$  is a diagonal matrix with unit operators  $I$  on diagonal. Using lemma 3.2 we obtain that

$$\| \|S^{-1}B\| \| \leq cn^{\gamma-1} \ln(n) \rightarrow 0, \quad n \rightarrow \infty.\tag{3.21}$$

It means that for  $n$  large enough there exists the matrix  $[E - S^{-1}B]^{-1}$  and

$$\| \| [E - S^{-1}B]^{-1} \| \| \leq c.$$

Consequently we obtain the following stability estimates from (3.20) using lemma 3.2:

$$\begin{aligned}\| \|\tilde{x}\| \| &\leq cn\|\phi\|, \\ \| \|\tilde{z}\| \| &\leq cn\|\psi\|.\end{aligned}\tag{3.22}$$

Let  $\Pi_n$  be the set of all polynomials in  $t$  with vector coefficients of degree less or equal then  $n$ . In complete analogy with [1, 23, 24] the following Lebesgue inequality for vector-valued functions can be proved

$$\| \|u(t) - P_n(t; u)\| \|_{C[-1,1]} \equiv \max_{t \in [-1,1]} \|u(t) - P_n(t; u)\| \leq (1 + \Lambda_n)E_n(u),\tag{3.23}$$

with the error of the best approximation of  $u$  by polynomials of degree not greater then  $n$

$$E_n(u) = \inf_{p \in \Pi_n} \max_{t \in [-1,1]} \|u(t) - p(t)\|.\tag{3.24}$$

Now, we can go over to the main result of this section.

**Theorem 3.3** *Let the assumptions of Lemma 3.2 with  $\gamma < 1$  hold, then there exists a positive constant  $c$  such that*

1. *For  $n$  large enough it holds*

$$\| \|\tilde{z}\| \| \leq cn^{\gamma-1} \cdot \ln n \cdot E_n(A_0^\gamma v),\tag{3.25}$$

*where  $v$  is the solution of (3.1);*

2. *The first equation in (3.17) can be written in the form*

$$\tilde{x} = S^{-1}B\tilde{x} + S^{-1}\phi,\tag{3.26}$$

*and can be solved by the fixed point iteration*

$$\tilde{x}^{(k+1)} = S^{-1}B\tilde{x}^{(k)} + S^{-1}\phi, \quad k = 0, 1, \dots; \quad \tilde{x}^{(0)} - \text{arbitrary},\tag{3.27}$$

*with the convergence rate of an geometrical progression with the denominator  $q \leq cn^{\gamma-1} \ln(n) < 1$  for  $n$  large enough.*

*Proof.* For  $\tilde{z}$  we have the second estimate in (3.22). The norm of the first summand on the right hand side of this inequality can be estimated in the following way

$$\begin{aligned}\| \|\psi\| \| &= \max_{1 \leq k \leq n} \left\| \int_{t_{k-1}}^{t_k} \left\{ A_k^\gamma e^{-A_k(t_k-s)} [A_k - A(s)] A_k^{-\gamma} (A_k^\gamma A_0^{-\gamma}) (A_0^\gamma v(s) - P_n(s; A_0^\gamma v)) \right\} ds \right\| \\ &\leq c \max_{1 \leq k \leq n} \int_{t_{k-1}}^{t_k} |t_k - s|^{-\gamma} |t_k - s| \|A_0^\gamma v(s) - P_n(s; A_0^\gamma v)\| ds \\ &\leq c\tau_{max}^{2-\gamma} \|A_0^\gamma u(s) - P_n(\cdot; A_0^\gamma v)\|_{C[-1,1]} \leq c\tau_{max}^{2-\gamma} (1 + \Lambda_n) E_n(A_0^\gamma v).\end{aligned}$$

So, we obtain

$$\| \|\psi\| \| \leq cn^{\gamma-2} \cdot \ln n \cdot E_n(A_0^\gamma u),\tag{3.28}$$

Now, the first assertion of the theorem follows from (3.22), (3.28). The second one follows from (3.17) and (3.21). ■

## 4 Examples

Let us consider the following problem

$$\begin{aligned} \frac{\partial u(x, t)}{\partial t} - \frac{\partial^2 u(x, t)}{\partial x^2} + q(x, t)u(x, t) &= f(x, t), \\ u(0, t) = u(1, t) &= 0, \\ u(x, -1) + \alpha u(x, 1) &= \varphi(x), \end{aligned} \tag{4.1}$$

with  $f(x, t) = e^{-\pi^2(1+t)} \sin(\pi x)(1+t)$ ,  $\alpha = 0.5$ ,  $\varphi(x) = \left(1 + 0.5e^{-2\pi^2}\right) \sin(\pi x)$ ,  $q(x, t) = 1+t$ . Then, the solution of this problem is  $u(x, t) = e^{-\pi^2(1+t)} \sin(\pi x)$ .

The problem (4.1) can be write down in the form (3.1) where the operator  $A(t)$  is defined by

$$\begin{aligned} D(A(t)) &= D(A) = \{v \in H^2(0, 1) : v(0) = 0, v(1) = 0\}, \\ A(t)v &= -\frac{\partial^2 v}{\partial x^2} + (1+t)v. \end{aligned} \tag{4.2}$$

Coefficients of the system (3.15) were calculated by using the Fourier series expansion. The results of calculation are presented in tables confirm our theory above.

Point $t$	$\varepsilon$
-1	0.00005276
-0.70710678	0.00097645
0	0.00063440
0.70710678	0.00029592
1	0.00010552

Table 4.1: The error in the case  $n = 4$ ,  $x = 0.5$

Point $t$	$\varepsilon$
-1	8.12568908Ee-7
-0.86602540	0.00010146
-0.5	0.00030932
0	0.00022136
0.5	0.00013419
0.86602540	0.00007182
1	0.00000162

Table 4.2: The error in the case  $n = 6$ ,  $x = 0.5$

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## References

- [1] K.I. Babenko. *Foundations of the numerical analysis (in Russian)*. Nauka, Moscow, 1986.
- [2] L. Byszewski. Theorems about the existence and uniqueness of solution of a semilinear evolution nonlocal cauchy problem. *J. Math. Anal. Appl.*, 162(5):494–505, 1991.
- [3] L. Byszewski. Application of properties of the right-hand sides of evolution equations to an investigation of nonlocal evolution problems. *Nonlinear Anal.*, 33(5):413–426, 1998.

Point $t$	$\varepsilon$
-1	0.00000117
-0.92387953	0.00000613
-0.70710678	0.00004544
-0.38268343	0.00005753
0	0.00004745
0.38268343	0.00003362
0.70710678	0.00002096
0.92387953	0.00000846
1	0.00000235

Table 4.3: The error in the case  $n = 8$ ,  $x = 0.5$

Point $t$	$\varepsilon$
-1	0.49451310e-8
-0.96592582	0.14687232e-7
-0.86602540	0.23393074e-6
-0.70710678	0.54494052e-6
-0.5	0.76722515e-6
-0.25881904	0.82803283e-6
0	0.76362937e-6
0.25881904	0.63174173e-6
0.5	0.47173110e-6
0.70710678	0.30381367e-6
0.86602540	0.14341583e-6
0.96592582	0.21271757e-7
1	0.98902621e-8

Table 4.4: The error in the case  $n = 12$ ,  $x = 0.5$

- [4] Ph. Clement, H.J.A.M. Heijmans, S. Angenent, C.J. van Duijn, and B. de Pagter. *One-parameter semigroups*. CWI Monographs, 5. North-Holland Publishing Co., Amsterdam, 1987.
- [5] M.L. Fernandez, Ch. Lubich, C. Palencia, and A. Schädle. Fast Runge-Kutta approximation of inhomogeneous parabolic equations. *Numerische Mathematik*, pages 1–17, 2005.
- [6] H. Fujita, N. Saito, and T. Suzuki. *Operator Theory and Numerical Methods*. Elsevier, Heidelberg, 2001.
- [7] I.P. Gavrilyuk, W. Hackbusch, and B.N. Khoromskij. Data-sparse approximation of a class of operator-valued functions. *Math. Comp.*, 74:681–708, 2005.
- [8] I.P. Gavrilyuk, W. Hackbusch, and B.N. Khoromskij. Tensor-product approximation to elliptic and parabolic solution operators in higher dimensions. *Computing*, 74:131–157, 2005.
- [9] I.P. Gavrilyuk and V.L. Makarov. Exponentially convergent parallel discretization methods for the first order evolution equations. *Computational Methods in Applied Mathematics (CMAM)*, 1(4):333– 355, 2001.
- [10] I.P. Gavrilyuk and V.L. Makarov. Algorithms without accuracy saturation for evolution equations in Hilbert and Banach spaces. *Math. Comp.*, 74:555–583, 2005.
- [11] I.P. Gavrilyuk and V.L. Makarov. Exponentially convergent algorithms for the operator exponential with applications to inhomogeneous problems in Banach spaces. *SIAM Journal on Numerical Analysis*, 43(5):2144–2171, 2005.



Point $t$	$\varepsilon$
-1	0.20628738e-11
-0.98078528	0.28602854e-10
-0.92387953	0.48425552e-9
-0.83146961	0.14258845e-8
-0.70710678	0.25968220e-8
-0.55557023	0.36339719e-8
-0.38268343	0.42916820e-8
-0.19509032	0.44975339e-8
0	0.43045006e-8
0.19509032	0.38169887e-8
0.38268343	0.31414290e-8
0.55557023	0.23686579e-8
0.70710678	0.15787207e-8
0.83146961	0.85640040e-9
0.92387953	0.30309439e-9
0.98078528	0.16809109e-10
1	0.41257476e-11

Table 4.5: The error in the case  $n = 16$ ,  $X = 0.5$

- [12] D. Gordeziani and Avalishvili G. Investigation of the nonlocal initial boundary value problems for some hyperbolic equations. *Hiroshima Math. J.*, 31(3):345–366, 2001.
- [13] S.G. Krein. *Linear Differential Operators in Banach Spaces*. Amer. Math. Soc., New York, 1971.
- [14] A.W. Leung and Chen G.-S. Optimal control of multigroup neutron fission systems. *J. Appl. Math. and Opt.*, 40(1):39–60, 1999.
- [15] A.W. Leung and Ortega L.A. Existence and monotone scheme for time-periodic nonquasimonotone reaction-diffusion systems: application to autocatalytic chemistry. *J. Math. Anal. Appl.*, 221(2):712–733, 1998.
- [16] M. López-Fernández and C. Palencia. On the numerical inversion of the laplace transform of certain holomorphic mappings. *Applied Numerical Mathematics*, 51:289–303, 2004.
- [17] M. López-Fernández, C. Palencia, and A. Schädle. A spectral order method for inverting sectorial laplace transforms. *SIAM J. Numer. Anal.*, 44:1332–1350, 2006.
- [18] A. Pazy. *Semigroups of linear operator and applications to partial differential equations*. Springer Verlag, New York, Berlin, Heidelberg, 1983.
- [19] A.A. Samarskii. Some problems of the theory of differential equations. *Differentsial'nye Uravneniya (Differential equations)*, 16(11):1925–1935, 1980.
- [20] D. Sheen, I. H. Sloan, and V. Thomée. A parallel method for time-discretization of parabolic problems based on contour integral representation and quadrature. *Math. Comp.*, 69:177–195, 2000.
- [21] D. Sheen, I. H. Sloan, and V. Thomée. A parallel method for time-discretization of parabolic equations based on laplace transformation and quadrature. *IMA Journal of Numerical Analysis*, 23:269–299, 2003.
- [22] R. E. Showalter. Initial and final value problems for degenerate parabolic evolution systems. *Indiana Univ. Math. J.*, 28(6):883–893, 1979.
- [23] G. Szegő. *Orthogonal Polynomials*. American Mathematical Society, New York, 1959.
- [24] G. Szegő. *Orthogonal Polynomials (with an Introduction and a Complement by J.L.Geronimus) (in Russian)*. State Publishing House of Physical and Mathematical Literature, Moscow, 1962.

- [25] T.Ju.Bohonova, I.P.Gavrilyuk, V.L.Makarov, and V.B.Vasylyk. Exponentially convergent duhamel-like algorithms for differential equations with an operator coefficient possessing a variable domain in a banach space. *SIAM J. Numer. Anal.*, 46(5):365–396, 2007/08.
- [26] V. Vasylyk. Uniform exponentially convergent method for the first order evolution equation with unbounded operator coefficient. *Journal of Numerical and Applied Mathematics (ISSN 0868-6912)*, 1:99–104 (in Russian), 2003.
- [27] V. Vasylyk. Approximate solution of the cauchy problem for differential equation with sectorial operator in banach space (in ukrainian). *Bulletin of Lviv University. Physical and Mathematical sciences*, 9:34–46, 2004.