GENERAL METHODS GUIDELINES

The course paper in the subject *Theory of Mechanisms and Machines* is one of the basic kinds of the student's individual work. The purpose of the course paper is to enhance the knowledge acquired by the student in the lectures, practical classes and laboratory sessions, and develop the skills of making research and design of present-day aircraft mechanisms and machines.

The course paper is to include the following parts:

1. Kinematic and force analyses of a leverage.
2. A planetary gear design.
3. A gearing design.

Each part of the course paper should consist of a calculation and a graphical sections.

All calculation sections are to be presented as an explanatory note that is either typed or hand written in blue or black ink on one side of size A4 paper. Every sheet is to be paginated and have the following margins: top – 5 mm, bottom – 5 mm, right – 5 mm, and left – 20 mm.

An explanatory note should have the following four parts: a kinematic analysis of a leverage, a force analysis of a leverage, a planetary gearing design and a gearing design, as well as the contents table, the assignment, the list of literature used in working on the course paper. Each new part is to begin with a new page.

Each part must be subdivided into items marked with numerals separated by a point. The first numeral represents the number of the part, the second – shows the number of the item.

Calculations should be made in an order that corresponds to the graphical plots. All magnitudes that are part of formulas are to be explained. In addition, it is necessary to denote units of measurement of parameters calculated.

The graphical section is to be executed on size A1 whatman paper (part1) and A2 (part 2 and part 3) in pencil. Above every drawing there should be an inscription indicating the scale. The title block should be drawn in the bottom right hand corner.

1. KINEMATIC ANALYSIS OF LEVERAGES

1.1. Theoretical information

The kinematic analysis of a mechanism is carried out taking into account the time factor only. In this case all forces acting on mechanism links are ignored.

As we know from theoretical mechanics, the motion of any body is characterized by its translation in space, velocity and acceleration. That is why the main tasks of kinematic analysis are to plot mechanism diagrams and paths of motion of separate points, to determine mechanism extreme positions, linear velocities and accelerations of a mechanism points, angular velocities and accelerations of mechanism links, the radius of curvature at any path point and so on.

As a result of this analysis we can determine the correspondence of kinematic parameters (translations, velocities and accelerations) to predetermined conditions of mechanism functioning as well as receive initial data for making further calculations. The knowledge of kinematic parameters is necessary to determine dynamic loads (inertia forces, moments of a couple of inertia forces), mechanism kinetic energy and power. Paths of motion of some points and mechanism extreme positions help to determine the links relative positions during their motion, to eliminate their possible collisions, to determine the working stroke of links, etc.

There exist graphical, analytical and experimental methods of kinematic analysis.

Graphical research methods are most common in engineering. They are quite simple, clear and accurate for engineering calculations.

While using graphical methods it is necessary to plot either velocity and acceleration diagrams or kinematic diagrams.

Velocity and acceleration diagrams are drawn on the basis of vector equations that connect velocities and accelerations of separate points of mechanism links. These diagrams allow finding momentary velocities and accelerations of different mechanism points as well as angular velocities and accelerations of mechanism links.

Kinematic diagrams are graphs of mechanism point or link translation, velocity and acceleration depending upon the time or turning angle of an initial link. By means of these diagrams we may analyze the changes in the above mentioned kinematic parameters during a complete cycle of mechanism motion.

Analytical research methods are more complicated, but owing to the emergence of computers, they are widely practiced. These methods help to get multivariant solutions and to choose mechanism diagrams and sizes of their links providing optimum working conditions.

Experimental methods are used in experimental research work.

In graphical methods plotting mechanism diagrams, velocity and acceleration diagrams is made to a certain scale. For this purpose, the scale factor is employed. The <u>scale factor</u> is a ratio of a physical magnitude (length, velocity, acceleration, etc.) to a segment length that represents this magnitude in the figure. The scale factor is marked by letter K with the magnitude index that is in the figure. For example, the scale factor of length is marked by K_l , the scale factor of velocities is K_V, the scale factor of accelerations is K_a. A scale factor has a dimension, where the dimension of a real physical magnitude is the numerator and the dimension of length in millimeters is the denominator. Thus, the scale factor μ_l has the dimension of $\frac{\text{m}}{\text{mm}}$, μ_V has $\frac{\text{m/sec}}{\text{mm}}$ and so on. If, for example, the velocity of point A V_A = 5 m/sec is shown by segment $\overline{pa}$ = 100 mm in the figure, then the scale factor of the velocity diagram is

$$\mu_V = \frac{V_A}{\overline{pa}} = \frac{5\ \text{m/sec}}{100\ \text{mm}} = 0.05\frac{\text{m/sec}}{\text{mm}}.$$

In the figure, where this velocity is shown, the following inscription should be made: 1 mm ^ 0.05 m/sec. It means that in the figure 1mm corresponds to 0.05 m/sec.

It is necessary to note that the dimensions in all figures given in the guide were diminished. That is why the length of any segment in a figure is less than the corresponding length given in the guide text.

The order of kinematic analysis is determined by mechanism structure and depends upon the order of attachment of Assur’s group to a group of initial links. That is why mechanism structural analysis always precedes kinematic analysis. It allows reducing kinematic analysis of any mechanism to studying separate Assur's groups, for which special expedients of kinematic and force analyses have been developed.

Kinematic analysis should be carried out in the order opposite to mechanism structure.

After determining a mechanism structure we should plot a mechanism kinematic diagram.

The kinematic diagram of a mechanism is graphic representation of links relative position corresponding to a certain point of time and taking into account the scale. By means of a mechanism kinematic diagram we may analyze motion of both the whole mechanism and its separate links, plot path of motion of any mechanism point, find extreme positions of the mechanism.

Plotting a kinematic diagram of a mechanism is usually begun with drawing a link whose position is given for a predetermined instant of time. As a rule, it is the mechanism's initial link. Kinematic diagrams are drawn with the use of diagrammatic representations of links and kinematic pairs.

Basic equations of velocities and accelerations

According to the graphical method of kinematic analysis after drawing a mechanism kinematic diagram it is necessary to plot the velocity and the acceleration diagram. For that, we should set up vector equations of velocities and accelerations. We will consider two cases: when two points are parts of one link and when two points are parts of different links.

Two points being parts of one link. Let points A and B, which are removed relative to each other at distance l_{AB}, be parts of one link (Fig. 1).

According to theoretical mechanics the velocity of any point of a perfectly rigid body can be determined as the geometric sum of velocities of transportation and relative motions. In our case, the transportation motion is the motion of point A and the relative one is the rotatory motion of the link about point A. Taking into account this fact, we may set up a vector equation for finding the velocity of point B:

$$\overline{V}_B = \overline{V}_A + \overline{V}_{BA},$$

where V_{BA} is the velocity of point B with respect to point A.

In general, the vector of relative velocity is directed along the tangent to the path of motion of the corresponding point. In our case, $\overline{V}_{BA}$ is perpendicular to AB.

The magnitude of relative velocity V_{BA} is determined by the following formula:

$$V_{BA} = \omega \cdot \ell_{AB},$$

where ω is the link angular velocity.

When the direction of relative velocity $\overline{V}_{BA}$ is known we may find the direction of angular velocity ω and vice versa. For this purpose it is necessary to show the direction of V_{BA} at point B. The direction of ω is determined according to the direction of V_{BA}.

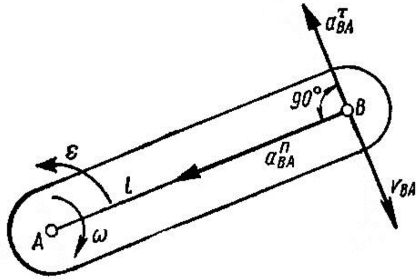


Fig.1.Relative motions of two points that are a part of one link

In the same way, we can write a vector equation for finding the acceleration of point B.

$$\overline{a}_B = \overline{a}_A + \overline{a}_{BA},$$

where a_{BA} is the acceleration of point B relative to point A.

During relative rotatory motion acceleration a_{BA} is resolved into normal acceleration a^n_{BA} directed to the centre of rotation, i.e. from point B to point A and tangential acceleration a^{τ}_{BA} directed along the tangent to the path of motion of point B (in our case perpendicular to AB). Then we obtain the following vector equation for point B

$$\overline{a}_B = \overline{a}_A + \overline{a}^n_{BA} + \overline{a}^{\tau}_{BA}.$$

The magnitudes of accelerations a^n_{BA} and a^{τ}_{BA} may be found according to the following formulas:

$$a^n_{BA} = \ell_{AB} \cdot \omega^2 = \frac{V^2_{BA}}{\ell_{AB}},$$

$$a_{BA}^{\tau} = \varepsilon \cdot \ell_{AB},$$

where ε is the link angular acceleration.

According to the direction of a_{BA}^{τ} we may determine the direction of ε and vice versa (Fig.1).

Two points belong to different links that form a sliding kinematic pair and coincide with each other at given instant of time. Let point A be a part of link 1 and point B belong to link 2 (Fig. 2). Links 1 and 2 form the sliding kinematic pair with the guide of motion H_{12}. In the same way as in the previous case, the velocity of point B consists of transportation and relative velocities. Transportation motion is the translational motion of link 1 and transportation velocity is the velocity of the point of link 1 that coincides with point B at a given instant of time (in our case it is point A). The relative velocity of point B is the velocity of link 2 relative to link 1. During the motion of link 2 with respect to link 1 point B moves along a straight line that is parallel to guide H_{12}. That is why relative velocity V_{BA} is parallel to H_{12}. Thus, the vector equation for the velocity of point B has the following form:

$$\overline{V}_B = \overline{V}_A + \overline{V}_{BA}.$$

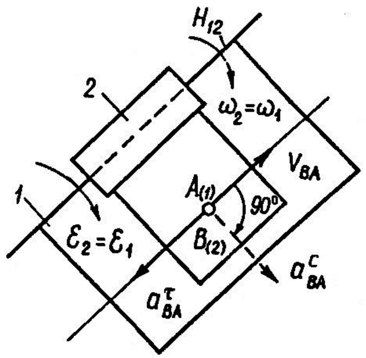


Fig.2. *R*elative motions of points A and B that belong to different links and coincide with each other at a given instant of time

The acceleration of point B consists of three components such as transportation acceleration (in our case it is the acceleration of point A) a_A, relative acceleration a_{BA}^{τ} and Coriolis acceleration a_{BA}^{c}. Then the vector equation for the acceleration of point B is

$$\overline{a}_B = \overline{a}_A + \overline{a}_{BA}^{c} + \overline{a}_{BA}^{\tau}.$$

Links 1 and 2, which form the sliding kinematic pair, do not have relative rotation. That is why their angular velocities and accelerations are identical, i.e.$\omega_2 = \omega_1$ and $\varepsilon_2 = \varepsilon_1$.

Coriolis acceleration a^{c}_{BA} appears as a result of interaction of transportation and relative motions. It may be found according to the following formula:

$$a^{c}_{BA} = 2 \cdot V_{BA} \cdot \omega_1 ,$$

where V_{BA} is the velocity of relative motion; ω_1 is the angular velocity of transportation motion.

Vector a^{c}_{BA} is directed to the side where vector V_{BA} will be directed if it is turned by 90° in the direction of angular velocity ω_1 (Fig. 2).

1.2. Kinematic analysis of the aircraft air compressor mechanism

Initial data

Geometrical dimensions of the mechanism: l_{OA} = 20 mm; l_{AB} = 60 mm; l_{AC} = 50 mm; l_{AS2} = 20 mm; l_{AS4} = 25 mm; angle between the guides Θ = 90˚. Link 1 rotates clockwise with constant rotational speed n_1 = 2000 rpm. Make a kinematic analysis of the mechanism at position #10 (Fig. 3,a).

1.2.1 Determination of the mechanism structure

In order to determine the mechanism structure it is necessary to find its degree of freedom.

Taking into account that the number of the movabel mechanism links n = 5, the number of the 1st kind kinematic pairs p_1 = 7, the number of the 2nd kind kinematic pairs p_2 = 0 the mechanism's degree of freedom W is calculated according to Chebyshev's formula

$$W=3\cdot n - 2\cdot p_1 - p_2 = 3\cdot 5 - 2\cdot 7 = 1.$$

The mechanism structure has the following form:

1. Links 4 and 5 make up dyad # 2.
2. Links 2 and 3 make up dyad #2.
3. Links 1 and 6 make up the group of initial links.

1.2.2. Plotting the mechanism's kinematic diagram

First, we should set the length of the segment that represents the greatest mechanism link. In our case it is link 2, connecting rod AB (Fig.3,*a*).

Let segment $\overline{AB}$ be equal to 240 mm (the length of this segment is chosen arbitrarily but it must be greater than 200 mm). Then the scale factor of the mechanism diagram is determined as

$$\mu_\ell = \frac{\ell_{AB}}{\overline{AB}} = \frac{0.06}{240} = 0.00025 \quad \frac{m}{mm}.$$

<u>Notes</u>. The length of segment $\overline{AB}$ should be taken thus in order to obtain a finite magnitude of the scale factor.

If the scale factor is known, we can find the length of segments that represent links 1 and 4.

$$\overline{AO} = \frac{\ell_{AO}}{\mu_\ell} = \frac{0.02}{0.00025} = 80 \ \ mm,$$

$$\overline{AC} = \frac{\ell_{AC}}{\mu_\ell} = \frac{0.05}{0.00025} = 200 \ \ mm.$$

Determine the length of segments $\overline{AS_2}$ and $\overline{AS_4}$:

$$\overline{AS_2} = \frac{\ell_{S2}}{\mu_\ell} = \frac{0.02}{0.00025} = 80 \ \ mm,$$

$$\overline{AS_4} = \frac{\ell_{S4}}{\mu_\ell} = \frac{0.025}{0.00025} = 100 \ \ mm.$$

It is necessary to note that in the figure ℓ_{OA}, ℓ_{AB}, ℓ_{AS2} are the real sizes of the links in meters and distances $\overline{AO}$, $\overline{AB}$, $\overline{AC}$, $\overline{AS_2}$ and $\overline{AS_4}$ are the lengths of the segments in millimeters.

Now let us switch over to plotting a kinematic diagram of the mechanism. First, we show the initial link (crank 1) in the given position and draw guides H_{36} and H_{50} through point O taking into account that angle between them is $\theta = 90^\circ$. In order to determine the positions of points B and C we will use the intersection method. According to this

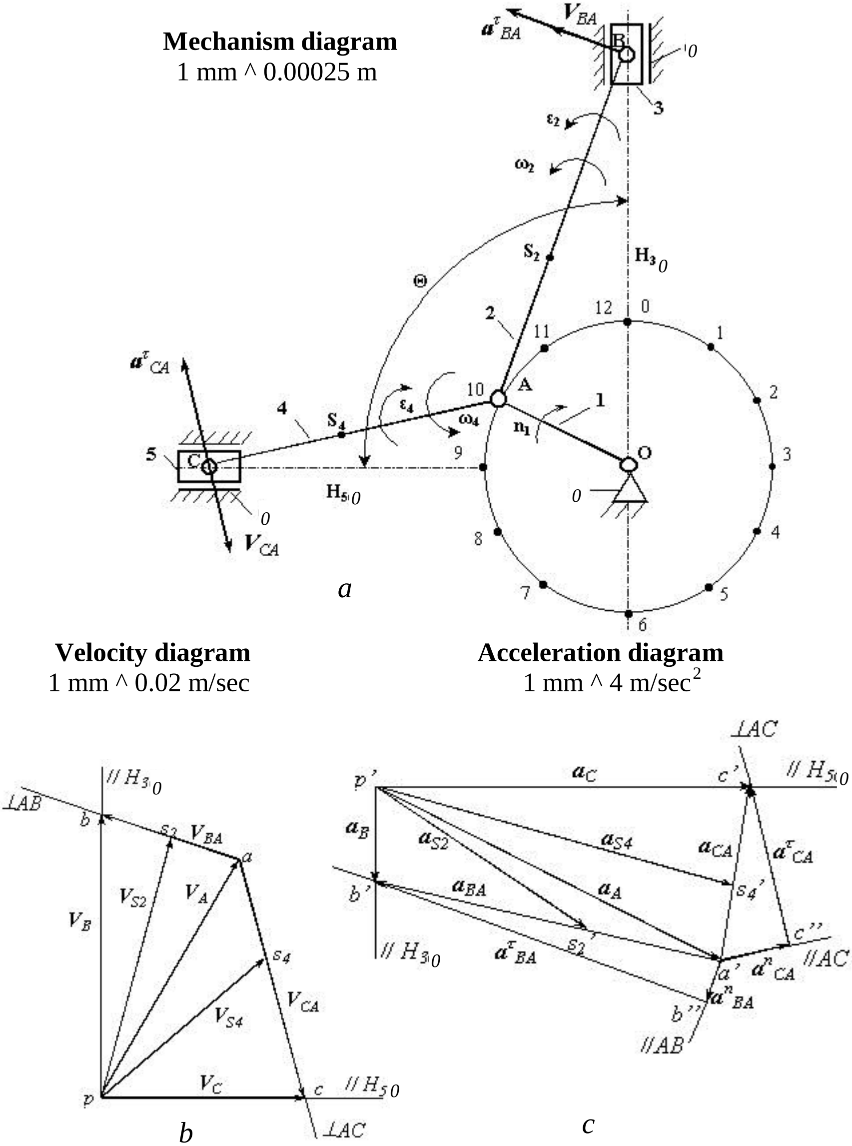


Fig.3.Kinematic analysis of the aircraft air compressor mechanism:
a – mechanism diagram; *b* – velocity diagram; *c* – acceleration diagram

method we should strike arcs of radii AB and AC on the guides H_{36} and H_{50}. After joining points B and C with point A and using corresponding links diagrammatic representations we obtain a diagram of the mechanism in position #10. The positions of points S_2 and S_4 can be found in the same way as for points B and C.

1.2.3. Plotting the velocity diagram

1. Determine the velocity of the initial link of the mechanism the motion of which is characterized by the motion of point A. As the initial link is hinged with the fixed link the velocity of point A is determined according to the following vector equation

$$\overline{V}_A = \overline{V}_O + \overline{V}_{AO},$$

where $\overline{V}_O$ is the velocity of the hinge centre O relative to which point A moves; $\overline{V}_{AO}$ is the relative velocity of point A during its motion with respect to point O.

Since point O is a fixed one $\overline{V}_O = 0$ and, consequently, $\overline{V}_A = \overline{V}_{AO}$.

Let us analyze the velocity of relative motion $\overline{V}_{AO}$. As link 1 rotates with the constant rotational speed, we can determine the magnitude of $\overline{V}_{AO}$ according to Euler's formula:

$$V_{AO} = \omega_1 \cdot \ell_{OA} = \frac{\pi \cdot n_1}{30} \cdot \ell_{OA} = \frac{3.14 \cdot 2000}{30} \cdot 0.02 = 4.2 \text{ m/sec.}$$

In general, the relative velocity is directed along the tangent to the path of motion of the corresponding point. Taking into account the fact that the path of motion of point A is a circle of radius AO, we can make a conclusion that $\overline{V}_{AO}$ is perpendicular to $\overline{AO}$ segment.

Now we will plot a velocity diagram for link 1. For this purpose it is necessary to give the length of segment $\overline{pa}$ that represents the velocity $\overline{V}_{AO}$ in the figure. Let $\overline{pa} = 210$ mm. Then the scale factor of the velocity diagram is:

$$\mu_V = \frac{V_A}{\underline{\underline{\mathbf{pa}}}} = \frac{4.2}{210} = 0.02 \quad \frac{m/\sec}{mm}.$$

Notes. The length of segment $\overline{pa}$ should be taken thus in order to obtain a finite magnitude of the scale factor.

Let us choose arbitrary point p (the pole of the velocity diagram) and lay off segment $\overline{pa}$ perpendicular to $\overline{\mathrm{OA}}$ in the direction of crank rotation (Fig. 3, *b*).

2. Determine the velocity of internal kinematic pair B of dyad #2 consisting of links 2 and 3. For this purpose we should set up the following vector equation

$$\underline{\overline{\mathrm{V}}_{\mathrm{B}}} = \underline{\underline{\overline{\mathrm{V}}_{\mathrm{A}}}} + \underline{\overline{\mathrm{V}}_{\mathrm{BA}}},$$

where $\overline{\mathrm{V}}_{\mathrm{B}}$ is the velocity of point B; $\overline{\mathrm{V}}_{\mathrm{BA}}$ is the velocity of relative motion during rotation of point B with respect to point A.

Let us analyse all components of the obtained equation. Velocity $\overline{\mathrm{V}}_{\mathrm{A}}$ was found both by magnitude and direction. The velocity of relative motion $\overline{\mathrm{V}}_{\mathrm{BA}}$ is known to us by direction only. It is perpendicular to $\overline{\mathrm{BA}}$. As slider 3 moves along fixed guide H_{36} velocity $\overline{\mathrm{V}}_{\mathrm{B}}$ is parallel to the latter.

Thus, we have vector equation with two unknowns (the magnitudes of vectors $\overline{\mathrm{V}}_{\mathrm{BA}}$ and $\overline{\mathrm{V}}_{\mathrm{B}}$). It can be solved by the graphical method. According to this method we should pass a straight line through point a that shows the direction of relative velocity $\overline{\mathrm{V}}_{\mathrm{BA}}$ ($\perp$BA) and a line parallel to guide H_{30} from pole p (Fig.3, *b*). The point of intersection of these lines indicates the position of point *b*. Segment $\overline{pb}$ represents velocity $\overline{\mathrm{V}}_{\mathrm{B}}$ to scale and segment $\overline{ab}$ shows velocity $\overline{\mathrm{V}}_{\mathrm{BA}}$. The direction of these velocities is determined according to the rule of vector compositions. In our case vector $\overline{\mathrm{V}}_{\mathrm{B}}$, as well as $\overline{\mathrm{V}}_{\mathrm{BA}}$, tend to point *b*. In order to find the magnitudes of velocities $\overline{\mathrm{V}}_{\mathrm{B}}$ and $\overline{\mathrm{V}}_{\mathrm{BA}}$, we should multiply corresponding lengths of the segments by scale factor μ_V.

$$V_B = \mu_V \cdot \overline{pb} = 0.02 \cdot 230 = 4.6 \text{ m/sec},$$

$$V_{BA} = \mu_V \cdot \overline{ab} = 0.02 \cdot 120 = 2.4 \text{ m/sec}.$$

3. Determine the velocity of point S_2 located on link 2. For this purpose we should use the similarity theorem. According to this theorem, point S_2 is to be located in the velocity diagram in the same way as in the mechanism diagram, i.e. between points *a* and *b*. In order to determine the position of point S_2 it is necessary to set up the following proportion

$$\frac{\overline{as_2}}{\overline{\overline{AS_2}}} = \frac{\overline{ab}}{\overline{\overline{AB}}},$$

whence

$$\overline{as_2} = \frac{\overline{AS_2} \cdot \overline{ab}}{\overline{\overline{AB}}} = \frac{80 \cdot 120}{240} = 40 \text{ mm}.$$

Let us lay off segment $\overline{as_2}$ along segment $\overline{ab}$. After connecting point s_2 with pole *p* we obtain segment $\overline{ps_2}$ (Fig.3, *b*) that represents the velocity of point S_2 to scale. The magnitude of this velocity is determined as

$$V_{S2} = \mu_v \cdot \overline{ps_2} = 0.02 \cdot 170 = 3.4 \text{ m/sec}.$$

4. Determine the velocity of point C that characterizes Assur's group consisting of links 4 and 5. The velocity of this point is determined according to the following vector equation

$$\underline{\overline{V}_C} = \underline{\underline{\overline{V}_A}} + \underline{\overline{V}_{CA}},$$

where $\overline{V}_C$ is the velocity of point C; $\overline{V}_{CA}$ is the velocity of relative motion during rotation of point C relative to point A.

The velocity of point A $\overline{V}_A$ was found both by magnitude and direction. The velocity of relative motion $\overline{V}_{CA}$ is perpendicular to $\overline{AC}$. The magnitude of this velocity should be found as well as the magnitude of velocity $\overline{V}_C$. The direction of the latter is parallel to H_{50}.

Let us plot the velocity diagram for considered Assur's group. According to the vector equation we pass a straight line through point *a* that is perpendicular to $\overline{AC}$ and a line parallel to guide H_{50} through pole *p*. Point *c* is the intersection of these lines. Segment $\overline{pc}$ represents velocity $\overline{V}_C$ to scale and segment $\overline{ac}$ characterizes relative velocity $\overline{V}_{CA}$. According to the rule of vector compositions velocities $\overline{V}_C$ and $\overline{V}_{CA}$ are directed to point *c*. In order to find the magnitudes of velocities $\overline{V}_C$ and $\overline{V}_{CA}$, it is necessary to multiply corresponding lengths of segments by scale factor K_V.

$$V_C = \mu_V \cdot \overline{pc} = 0.02 \cdot 150 = 3 \text{ m/sec},$$

$$V_{CA} = \mu_V \cdot \overline{ac} = 0.02 \cdot 205 = 4.1 \text{ m/sec}.$$

5. Determine the velocity of point S_4. Since point S_4 is located on segment $\overline{AC}$ in the mechanism diagram, it should be located on the identical segment $\overline{ac}$ of the velocity diagram. The disposition of this point is determined from the following proportion

$$\frac{\overline{as_4}}{\overline{\overline{AS_4}}} = \frac{\overline{ac}}{\overline{\overline{AC}}},$$

whence

$$\overline{as_4} = \frac{\overline{AS_4} \cdot \overline{ac}}{\overline{\overline{AC}}} = \frac{100 \cdot 205}{200} = 102.5 \text{ mm}.$$

After laying off segment $\overline{as_4}$ along segment $\overline{ac}$ and connecting point s_4 with pole *p*, we obtain segment $\overline{ps_4}$ that represents velocity $\overline{V}_{S4}$. The magnitude of this velocity is determined as

$$V_{S4} = \mu_V \cdot \overline{ps_4} = 0.02 \cdot 160 = 3.2 \text{ m/sec}.$$

6. Determine the angular velocities of links 2 and 4. The magnitudes of these angular velocities are determined according to the following formulas

$$\omega_2 = \frac{V_{BA}}{\ell_{AB}} = \frac{2.4}{0.06} = 40 \ \frac{\text{rad}}{\text{sec}},$$

$$\omega_4 = \frac{V_{CA}}{\ell_{AC}} = \frac{4.1}{0.05} = 82 \frac{\text{rad}}{\text{sec}}.$$

In order to determine the directions of angular velocities of links 2 and 4, it is necessary to transfer the vectors of relative velocities $\overline{V}_{BA}$ and $\overline{V}_{CA}$ in the velocity diagram to corresponding points B and C in the mechanism diagram and to consider the motion of points B and C relative to point A. In our case, angular velocities ω_2 and ω_4 are anti-clockwise (Fig.3, *a*).

1.2.4. Plotting the acceleration diagram

1. Determine the acceleration of the mechanism initial link the motion of which is characterized by the motion of point A. The acceleration of point A that performs a rotatory motion along the circle of radius ℓ_{OA} is determined according to the following vector equation

$$\overline{a}_A = \overline{a}_O + \overline{a}_{AO}^{n} + \overline{a}_{AO}^{\tau},$$

where $\overline{a}_O$ is the acceleration of hinge centre O relative to which point A moves; $\overline{a}_{AO}^{n}$ is the normal acceleration of point A relative to point O; $\overline{a}_{AO}^{\tau}$ is the tangential acceleration of point A relative to point O.

Since point O is a fixed one a_O = 0. Normal acceleration $\overline{a}_{AO}^{n}$ is directed to the centre of rotation of point A, i.e. to point O. The magnitude of this acceleration is determined as

$$a_{AO}^{n} = \omega_1^2 \cdot \ell_{OA} = \left(\frac{\pi \cdot n_1}{30}\right)^2 \cdot \ell_{OA} = \left(\frac{3.14 \cdot 2000}{30}\right)^2 \cdot \ 0.02 = 876 \ \frac{\text{m}}{\text{sec}^2}.$$

Tangential acceleration $\bar{a}_{AO}^{\tau}$ is directed along the tangent to the path of motion of point A and is determined by the following formula

$$a_{AO}^{\tau} = \varepsilon_1 \cdot \ell_{\mathrm{OA}},$$

where ε_1 is the angular acceleration of the initial link. Since crankshaft 1 rotates with the constant angular velocity we can make a conclusion that $\varepsilon_1 = 0$ and consequently $\bar{a}_A = \bar{a}_{AO}^{n}$.

Let segment $\overline{p'a'}$ represent the acceleration of point A. The length of this segment is chosen arbitrarily. In our case, we assume $\overline{p'a'}$ =219 mm. Then the scale factor in the acceleration diagram is found as

$$\mu_a = \frac{a_A}{\overline{\overline{p'a'}}} = \frac{876}{219} = 4 \ \frac{\mathrm{m/sec^2}}{\mathrm{mm}}.$$

<u>Notes.</u> The length of segment $\overline{p'a'}$ should be taken in such a way to obtain a finite magnitude of the scale factor.

Let us choose arbitrary point *p'* (the pole in the acceleration diagram) and lay off segment $\overline{p'a'}$ parallel to $\overline{\mathrm{OA}}$ in the direction from point A to point O (Fig.3, *c*).

2. Determine the acceleration of point B that characterizes dyad #3 consisting of links 2 and 3. For this purpose we should set up the following vector equation

$$\underline{\bar{a}_B} = \underline{\underline{\bar{a}_A}} + \underline{\underline{\bar{a}_{BA}^{n}}} + \underline{\bar{a}_{BA}^{\tau}},$$

where $\bar{a}_B$ is the absolute acceleration of point B that is parallel to guide H_{30}; $\bar{a}_{BA}^{n}$ is the normal acceleration of point B relative to point A that is directed parallel to $\overline{\mathrm{AB}}$ from point B to point A; $\bar{a}_{BA}^{\tau}$ is the tangential acceleration of point B relative to point A that is perpendicular to $\overline{\mathrm{AB}}$.

Normal acceleration $\bar{a}_{BA}^{n}$ is determined as

$$a_{BA}^{n}=\frac{V_{BA}^{2}}{\ell_{BA}}=\frac{2.4^{2}}{0.06}=96\ \frac{m}{sec^{2}}.$$

Let us solve the vector equation mentioned above by the graphical method. For that, through point *a'* we draw straight line parallel to $\overline{AB}$ and in the direction from point B to point A in the mechanism diagram we lay off segment $\overline{a'b''}$ that represents normal acceleration $\bar{a}_{BA}^{n}$ to scale (Fig.3, *c*). The magnitude of this segment is determined by the following formula

$$\overline{a'b''}=\frac{a_{BA}^{n}}{\mu_a}=\frac{96}{4}=24\ \text{mm}.$$

Through the obtained point *b''* we pass straight line perpendicular to $\overline{AB}$ and through pole *p'* it necessary to draw a line parallel to H_{30}. At the point of intersection of these lines we obtain point *b'*. Segment $\overline{p'b'}$ represents the absolute acceleration of point B to scale and segment $\overline{b'b''}$ shows tangential acceleration $\bar{a}_{BA}^{\tau}$. The magnitudes of these accelerations can be found in the following way

$$a_B=\mu_a\cdot\overline{p'b'}=4\cdot 100=400\ \text{m/sec}^2,$$

$$a_{BA}^{\tau}=\mu_a\cdot\overline{b'b''}=4\cdot 200=800\ \text{m/sec}^2.$$

3. Determine the acceleration of point S_2 located on link 2. For this purpose we should use the similarity theorem. According to this theorem point S_2 is to be located in the acceleration diagram in the same way as in the mechanism diagram, i.e. between points *a'* and *b'*. In order to determine the position of point S_2 it is necessary to set up the following proportion

$$\frac{\overline{a's_2'}}{\overline{AS_2}}=\frac{\overline{a'b'}}{\overline{AB}},$$

whence

$$\overline{a's_2'}=\frac{\overline{AS_2}\cdot\overline{a'b'}}{\overline{AB}}=\frac{80\cdot 190}{240}=63.3\ \text{mm}.$$

Let us draw segment $\overline{a's_2'}$ along segment $\overline{a'b'}$. After connecting point s_2' with pole p' we obtain segment $\overline{p's_2'}$ that represents the acceleration of point S_2 to scale. The magnitude of this acceleration is determined as

$$a_{S2} = \mu_a \cdot \overline{p's_2'} = 4 \cdot 140 = 560 \text{ m/sec}^2.$$

4. Determine the acceleration of internal kinematic pair C of dyad #3 consisting of links 4 and 5. In this case we should set up the following vector equation

$$\underline{\bar{a}_C} = \underline{\underline{\bar{a}_A}} + \underline{\underline{\bar{a}_{CA}^n}} + \underline{\bar{a}_{CA}^\tau},$$

where $\bar{a}_C$ is the absolute acceleration of point C ($\bar{a}_C \parallel H_{50}$); $\bar{a}_{CA}^n$ is the normal acceleration of point C relative to point A that is directed parallel to $\overline{AC}$ from point C to point A; $\bar{a}_{CA}^\tau$ is the tangential acceleration of point C relative to point A ($\bar{a}_{CA}^\tau \perp AC$).

We can find the normal acceleration $\bar{a}_{CA}^n$ as

$$a_{CA}^n = \frac{V_{CA}^2}{\ell_{CA}} = \frac{4.1^2}{0.05} = 336.2 \ \frac{\text{m}}{\text{sec}^2}.$$

Let us represent this acceleration in the acceleration diagram by segment $\overline{a'c''}$ the length of which is

$$\overline{a'c''} = \frac{a_{CA}^n}{\mu_a} = \frac{336.2}{4} = 84.05 \text{ mm}.$$

Now, through point a' we draw a straight line parallel to $\overline{AC}$ in the direction from point C to point A in the mechanism diagram and lay off the length of segment $\overline{a'c''}$. After that through the obtained point c'' we pass a straight line perpendicular to $\overline{AC}$ and through pole p' we draw a line parallel to H_{50}. The point of intersection of these lines is point c' (Fig.3, *c*). Segment $\overline{p'c'}$ represents the absolute acceleration of point C to scale and segment $\overline{c'c''}$ denotes tangential acceleration $\overline{a}_{CA}^{\tau}$. The magnitudes of these accelerations are determined as

$$a_C = \mu_a \cdot \overline{p'c'} = 4 \cdot 195 = 780 \text{ m/sec}^2,$$

$$a_{CA}^{\tau} = \mu_a \cdot \overline{c'c''} = 4 \cdot 80 = 320 \text{ m/sec}^2.$$

5. Determine the acceleration of point S_4 located on link 4. According to the similarity theorem point S_4 is to be located in the acceleration diagram between points *a'* and *c'*. In order to determine the position of point S_4 we set up the following proportion

$$\frac{\overline{a's_4'}}{\overline{\overline{AS_4}}} = \frac{\overline{a'c'}}{\overline{\overline{AC}}},$$

whence

$$\overline{a's_4'} = \frac{\overline{AS_4} \cdot \overline{a'c'}}{\overline{\overline{AC}}} = \frac{100 \cdot 103}{200} = 51.5 \text{ mm}.$$

Let us lay off segment $\overline{a's_4'}$ along segment $\overline{a'c'}$. After connecting point s_4' with pole *p'* we obtain segment $\overline{p's_4'}$ that represents $\overline{a}_{S4}$ to scale. The magnitude of this acceleration is determined as

$$a_{S4} = \mu_a \cdot \overline{p's_4'} = 4 \cdot 160 = 640 \text{ m/sec}^2.$$

6. Determine the angular accelerations of mechanism links 2 and 4. The magnitudes of these angular accelerations are determined according to the following formulas

$$\varepsilon_2 = \frac{a_{BA}^{\tau}}{\ell_{AB}} = \frac{800}{0.06} = 13333.33 \ \frac{\text{rad}}{\text{sec}^2},$$

$$\varepsilon_4 = \frac{a_{CA}^{\tau}}{\ell_{AC}} = \frac{320}{0.05} = 6400 \ \frac{\text{rad}}{\text{sec}^2}.$$

In order to determine the direction of the angular accelerations of links 2 and 4 it is necessary to transfer the vectors of tangential accelerations $\bar{a}_{BA}^{\tau}$ and $\bar{a}_{CA}^{\tau}$ in the acceleration diagram to corresponding points B and C in the mechanism diagram and to consider motion of points B and C relative to point A. In our case, angular acceleration ε_2 is anti-clockwise and ε_4 is clockwise (Fig.3, *a*) .

2. FORCE ANALYSIS OF LEVERAGES

2.1.Theoretical information

The main task of mechanism force analysis is to determine the forces of links interaction, i.e. pressures in kinematic pairs as well as unknown external forces acting on the mechanism links. Knowledge of these forces is necessary for mechanism strength analysis, rigidity analysis, vibration-resistance analysis, wear-resistance analysis, for the calculation of durability, for determining friction losses of power, etc.

All forces that act on mechanism links are divided into the following groups:

1. Driving forces that make a mechanism move. They act on the side of links motion. That is why the work of driving forces is always positive. Examples of driving forces are electromagnetic forces, pressure of steam or gas, pressure of water or air, elastic forces of springs, etc.

2. Forces of resistance that act opposite to links motion. Work of these forces is always negative. They are subdivided into forces of useful resistance and forces of parasitic resistance. Forces of useful resistance are the forces of technological resistance to motion for the overcoming of which work is expended when an engineering process is made. An example of such forces may be the force of resistance to metal cutting, forces of aerodynamic resistance that act on airplane propellers and so on. Forces of parasitic resistance are the forces for the overcoming of which the additional work is expended. They include frictional forces in kinematic pairs as well as forces of resistance of working medium. These forces are quite small in comparison with all other forces acting on a mechanism links. That is why in our further calculations we will neglect them.

3. Gravities that develop due to links' interaction with the Earth. In separate parts of mechanism motion these forces may perform both positive and negative work. But the work of gravities per complete kinematic cycle is equal to zero because the points of their application move cyclically.

4. Forces of inertia that are the result of nonuniform motion of mechanism links. The work of these forces is equal to zero per cycle of mechanism motion.

5. Reacting forces that develop owing to links' interaction with each other in places of their contact, i.e. in kinematic pairs. These are the forces with which one link acts on the other during motion. According to Newton's third law reacting forces are always inverse. In a mechanism, the number of these forces is equal to the number of kinematic pairs. *R*eacting forces are considered as internal ones for the whole mechanism although for every separate link they are external forces. The work of reacting forces is never equal to zero because frictional forces in kinematic pairs are not equal to zero.

The purpose of mechanism force analysis is to determine reacting forces in all kinematic pairs and unknown external forces or moments acting on a mechanism links.

An unknown external force (moment) is determined from the state of equilibrium of a mechanism initial link and is called the balancing force (balancing moment). This force (moment) balances the action of all forces applied to the initial link and, consequently, to a mechanism as a whole. The number of balancing forces (balancing moments) depends upon the number of initial links.

As initial data for force analysis the law of motion of an initial link, links' dimensions, masses, moments of inertia with respect to the centre of mass and external forces should be given. These data allow finding forces of inertia needed for further calculations.

As it is known from theoretical mechanics, elementary forces of inertia of any link can be reduced to resultant force of inertia F_{in} applied at the center of mass of the link and to the resultant couple of inertia forces whose moment is M_{in}.

Force of inertia $\overline{F}_{in}$ is found as

$$\overline{F}_{in} = -m \cdot \overline{a}_s ,$$

where m is the link mass in kg; $\overline{a}_s$ is the vector of total acceleration of the centre of mass S of a link in m/sec^2.

Sign "-" shows that force of inertia $\overline{F}_{in}$ is always directed opposite to the acceleration of the point of application of this force.

Moment of a couple of inertia forces $\overline{M}_{in}$ is determined by the formula:

$$\overline{M}_{in} = -J_s \overline{\varepsilon} ,$$

where Js is the moment of inertia of a link relative to the axis that passes through the center of mass; ε is the angular acceleration of a link.

According to the formula, moment $\overline{M}_{in}$ is directed opposite to link angular acceleration ε.

Force analysis may be carried out by different methods. The simplest of them is the method in which forces of inertia are not taken into account. This method takes place when accelerations of links are quite small and we may neglect them. In this case, reacting forces in kinematic pairs are determined from equations of statics. That is why it is called the static method of force analysis that is used for low-speed machines and mechanisms.

In general, forces of inertia are not small and they should be taken into account during calculation. For this purpose we will use combined static and inertia force analysis. This method is based on the use of d'Alembert's principle. According to this principle, if, besides the external forces, we apply to mechanism links forces of inertia, we may consider the whole mechanism and its separate links as stationary and in a state of equilibrium. In this case unknown forces are found from equations of statics that are written on the basis of methods of statics.

As it is known from theoretical mechanics methods of statics may be used for statically determinate systems. The statically determinate mechanical system is a system in which the number of unknown parameters is equal to the number of equations of equilibrium.

In order to determine what kinematic chain of a mechanism is the statically determinate system, we will consider a plane kinematic chain that consists of n movable links and p_1 kinematic pairs of the 1^{st} kind. For every plane link we may set up three equations of equilibrium (two equations of forces and one equation of moments). As the kinematic chain contains n links, the total number of equilibrium equations is 3n.

Now let us determine the number of unknown elements. For this purpose we should analyze reacting forces that develop in kinematic pairs.

In general, any force is a vector that is characterized by its magnitude, direction and the point of application. If we neglect the forces of friction in kinematic pairs reacting forces will be directed along the general normal to contact surfaces. For a turning kinematic pair this normal passes through the hinge center. Then the point of application of the reacting force is known and the unknowns are the magnitude and the direction of this force. For a sliding kinematic pair the general normal is perpendicular to the guide of motion. Consequently, we know the direction of the reacting force and it is necessary to find the magnitude and the point of application.

Thus, one known and two unknown parameters characterize a reacting force that develops in the 1st kind kinematic pair. As a kinematic chain has p_1 the 1st kind kinematic pairs the number of unknown elements is $2p_1$.

A kinematic chain will be statically determinate when the number of unknown parameters is equal to the number of equilibrium equations

$$2 \cdot p_1 = 3 \cdot n$$

or

$$3 \cdot n - 2 \cdot p_1 = 0.$$

But this formula determines the number of degrees of freedom of Assur's group. Therefore, Assur's group is a statically determinate kinematic chain.

Thus, for carrying out a force analysis, a mechanism should be divided into structural groups (Assur's groups and a group of initial links), i.e. we have to determine the mechanism structure. The order of the force analysis is the same as the representation of the mechanism structure. In other words, we should begin our force analysis with the determination of reacting forces in the kinematic pairs of the most remote Assur's group relative to the group of initial links and to finish it by the analysis of the group of initial links.

The force analysis of a mechanism may be made by graphical or analytical methods. The graphical method is the most convenient because it is quite simple and clear.

Using the graphical method we should plot force diagrams for every Assur's group and for separate links. Force diagrams are graphical solution of vector equations of equilibrium and are drawn as a closed polygon of forces. One vector equation is equivalent to two scalar (algebraic) equations and consequently allows finding two unknown parameters.

In the analytical method of force analysis every vector equation is replaced by two scalar equations of force projections to the coordinate axes.

Sometimes there is no necessity to carry out the total force analysis of a mechanism as a result of which reacting forces in kinematic pairs are found. In this case the task is reduced to the determination of a balancing force, or a balancing moment, that is necessary, for example, when calculating engine power. For that, we will use the method of Zhukovsky's rigid lever. N. Zhukovsky showed that the equilibrium of any mechanism with one degree of freedom corresponds to the equilibrium of any lever and proposed the following theorem.

If vectors of all forces applied at different link points of a mechanism are transferred parallel to themselves to the corresponding points of the velocity diagram that is turned through 90°, we can consider this velocity diagram as a rigid lever rotated around the pole and loaded by the same forces as a prime mechanism. The sum of moments of all forces acting on the rigid lever with respect to the pole will be equal to zero, i.e. $\sum M_p = 0$. By solving this equation we may find a balancing force.

Thus, to determine a balancing force (moment) by Zhykovsky's method, we have to plot a velocity diagram to an arbitrary scale that is turned through 90°, and to apply all forces that act on mechanism links at corresponding points. This method is used for checking the correctness of the combined static and inertia force analysis of a mechanism. The difference between the balancing forces (balancing moments) obtained using the above mentioned methods should be less than 5 %.

2.2. Force analysis of the aircraft air compressor mechanism

Initial data

Geometrical dimensions of the mechanism: l_{OA} = 20 mm; l_{AB} = 60 mm; l_{AC} = 50 mm; l_{AS2} = 20 mm; l_{AS4} = 25 mm; the angle between the guides $\Theta = 90^{\circ}$. Masses of the mechanism links: m_2 = 0.5 kg; m_3 = 0.4 kg; m_4 = 0.45 kg; m_5 = 0.35 kg. Centres of mass of links 1, 2, 3, 4 and 5 are, correspondingly, at points O, S_2, B, S_4, C. Moments of inertia of con-rods: $J_{S2} = 0.005$ kg·m^2; $J_{S4} = 0.004$ kg·m^2. Compressed air pressures: P_3 = 2 kN; P_5 = 1 kN. Link 1 rotates clockwise with constant rotational speed n_1=2000 rpm. Make force analysis for the mechanism at position #10.

2.2.1. Determination of the mechanism structure

The structure of the aircraft air compressor mechanism (Fig 4, *a*) has the following form:

1. Links 4 and 5 form dyad # 2.
2. Links 2 and 3 form dyad # 2.
3. Links 1 and 6 form the group of initial links.

The mechanism force analysis is carried out in the following order: first, we should determine the forces in kinematic pairs of the most remote Assur’s group with respect to the group of initial links. In our case this group is formed by links 5 and 4. Then, the forces in kinematic pairs of Assur’s group formed by links 3 and 2 are found. Finally, we determine the forces in kinematic pairs of the group of initial links and the balancing moment applied to initial link 1.

2.2.2. Plotting the velocity diagram and the acceleration diagram

For the mechanism given position we plot the velocity diagram, the acceleration diagram (Fig.4, *b*, *c*) and determine accelerations of centre of mass of all mechanism links as well as angular accelerations of links that perform rotatory motion.

The accelerations of points B, C, S_2, S_4 were found:

$$a_B = \mu_a \cdot \overline{p'b'} = 4 \cdot 100 = 400 \text{ m/sec}^2,$$

$$a_C = \mu_a \cdot \overline{p'c'} = 4 \cdot 195 = 780 \text{ m/sec}^2,$$

$$a_{S2} = \mu_a \cdot \overline{p's_2'} = 4 \cdot 140 = 560 \text{ m/sec}^2,$$

$$a_{S4} = \mu_a \cdot \overline{p's_4'} = 4 \cdot 160 = 640 \text{ m/sec}^2.$$

The angular accelerations of links 2 and 4 are determined as

$$\varepsilon_2 = \frac{a_{BA}^{\tau}}{\ell_{\mathrm{AB}}} = \frac{\mu_a \cdot \overline{b''b'}}{\ell_{\mathrm{AB}}} = \frac{4 \cdot 200}{0.06} = 13333.33 \ \frac{\text{rad}}{\text{sec}^2},$$

$$\varepsilon_4 = \frac{a_{CA}^{\tau}}{\ell_{\mathrm{AC}}} = \frac{\mu_a \cdot \overline{c''c'}}{\ell_{\mathrm{AB}}} = \frac{4 \cdot 80}{0.05} = 6400 \ \frac{\text{rad}}{\text{sec}^2}.$$

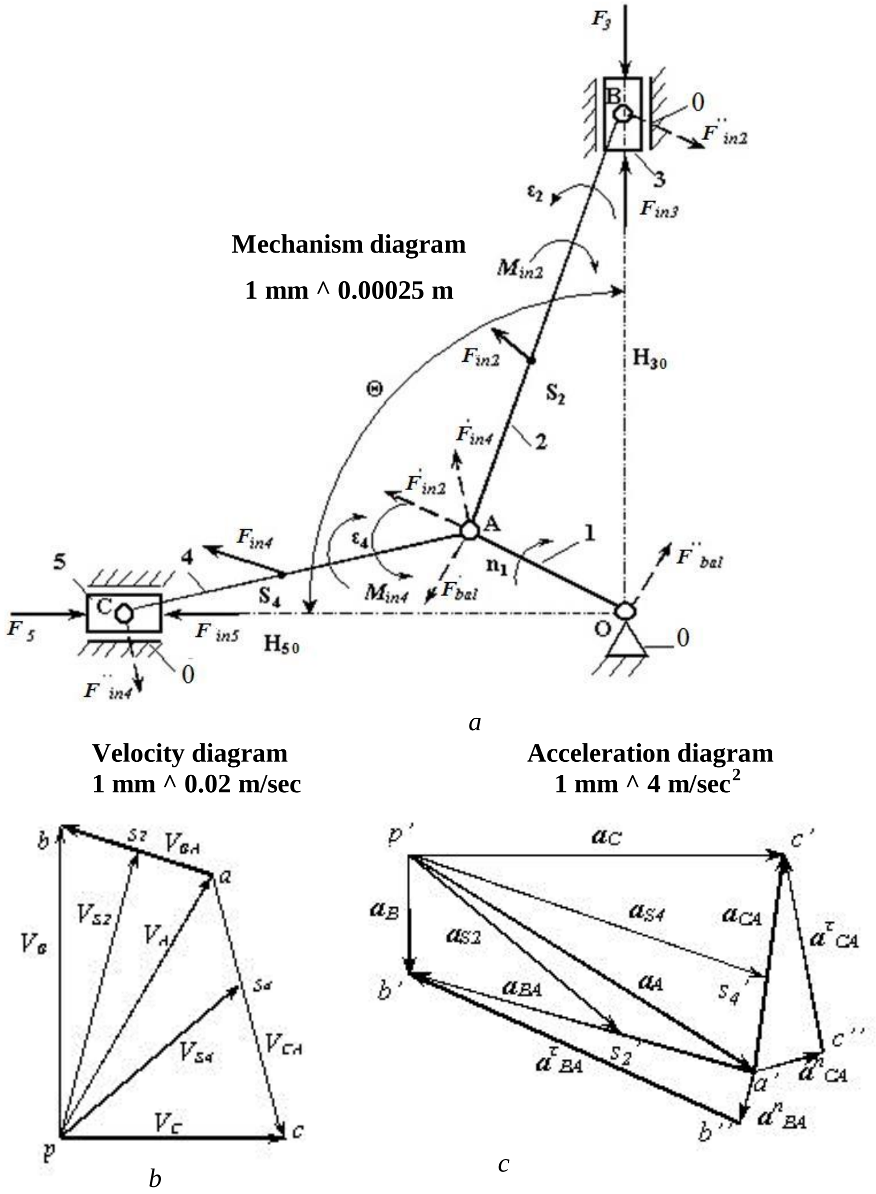


Fig.4. Determination of forces of inertia of the mechanism links:
a – mechanism diagram; *b* – velocity diagram; *c* – acceleration diagram

2.2.3. Determination of resultant forces of inertia and resultant moments of a couple of inertia forces

Magnitudes of links forces of inertia applied to corresponding centres of mass are determined by the following formulas:

$$F_{in2} = \mathrm{m}_2 \cdot a_{S2} = 0.5 \cdot 560 = 280 \text{ N},$$

$$F_{in3} = \mathrm{m}_3 \cdot a_B = 0.4 \cdot 400 = 160 \text{ N},$$

$$F_{in4} = \mathrm{m}_4 \cdot a_{S4} = 0.45 \cdot 640 = 288 \text{ N},$$

$$F_{in5} = \mathrm{m}_5 \cdot a_C = 0.35 \cdot 780 = 273 \text{ N}.$$

Forces of inertia are oppositely directed to accelerations of corresponding centres of mass of links (Fig. 4, a).

Magnitudes of moments of a couple of inertia forces that act on links 2 and 4 may be determined in the following way:

$$M_{in2} = \mathrm{J}_{\mathrm{S}2} \cdot \varepsilon_2 = 0.005 \cdot 13333.33 = 66.67 \text{ N·m},$$

$$M_{in4} = \mathrm{J}_{\mathrm{S}4} \cdot \varepsilon_4 = 0.004 \cdot 6400 = 25.6 \text{ N·m}.$$

The direction of these moments of a couple of inertia forces is opposite to the angular acceleration of the corresponding link.

In the combined static and inertia force analysis the obtained forces of inertia and the moments of a couple of inertia forces are considered as external forces that load the mechanism links. Let us show these forces in the mechanism diagram (Fig. 4, *a*).

2.2.4. Determination of reacting forces in kinematic pairs of Assur's group formed by links 4 and 5

We are going to determine the reacting forces in kinematic pairs of the mechanism ignoring the forces of friction that develop in these pairs and links' weight. It is explained by the fact that the forces mentioned above (the forces of friction and the weight of the links) have quite small magnitudes in comparison with the external forces acting on the mechanism links.

1. Plot Assur's group formed by links 4 and 5 at needed position taking into account the scale factor of length $\mu\ell = 0.00025\ \frac{\text{m}}{\text{mm}}$ (it was found when making the kinematic analysis of the mechanism) and apply all forces acting on the mechanism links. Besides, we will replace the action of separated links (in our case links 1 and 6) by reacting forces that develop in the corresponding kinematic pairs. The reacting force will be marked by letter *R* with double subscript of links that make up a kinematic pair. Dyad #2 contains three kinematic pairs. One of them formed by links 5 and 6 is a sliding pair. The two others formed by links 4 and 5, 4 and 1 are turning ones. That is why there are three reacting forces in the kinematic pairs such as R_{05}, R_{54}, R_{14}. It is necessary to take into consideration that $R_{05} = - R_{50}$, $R_{54} = - R_{45}$, $R_{14} = - R_{41}$. For this, Assur's group forces R_{05} and R_{14} are external ones and R_{54} is an internal force.

Let us analyze the reacting forces mentioned above. As links 5 and 6 make up a sliding kinematic pair, force R_{05} of interaction of these links is perpendicular to guide of motion H_{50}, but the magnitude and the point of application are unknown. We mark the line of action of this force with a dotted line perpendicular to H_{50} that passes at certain distance h_{05} relative to point C (Fig.5, *a*). Arm h_{05} determines the point of application of R_{05} and we are to find it.

Force R_{14} is a force that develops in turning kinematic pair A. That is why the point of application of this force is known (the center of the turning pair) and we should determine the magnitude and the direction of this force. As the direction of R_{14} is unknown, we will resolve it into two components: normal force R^{n}_{14} that is parallel to AC and tangential force R^{τ}_{14} that is perpendicular to AC (Fig.5, *a*). The direction of R^{τ}_{14} is chosen arbitrarily. If we obtain a negative magnitude of R^{τ}_{14}, the direction of this force was chosen incorrectly.

Thus, link 5 is loaded by four forces such as pressure F_5, inertia force F_{in5}, force R_{05} that acts from the side of the fixed link 6 and force R_{45} that acts from the side of link 4. Link 4 is under the action of one moment M_{in4} and three forces: inertia force F_{in4}, force R_{14} from the side of link 1 and force R_{54} from the side of link 5.

2. Determine tangential component R_{14}^{τ} of the reacting force, that develops in the turning pair. For this purpose we should consider the state of equilibrium of link 4. As this link is in a state of equilibrium the sum of moments of all acting forces relative to any point must be equal to zero. Let us set up an equation of moments with respect to point C ($\sum M_C = 0$). We assume that the clockwise moment is positive and the anticlockwise moment is negative.

$$R_{14}^{\tau} \cdot l_{AC} - F_{in4} \cdot h_4 - M_{in4} = 0$$

Forces R_{14}^{n} and R_{54} do not form moments because their arms relative to point C are equal to zero. Arm h_4 is determined from the Assur's group diagram (Fig.5, *a*). As h_4 is the shortest distance it is perpendicular to the line of action of force F_{in4}. In order to determine the real magnitude of this arm, it is necessary to multiply the corresponding segment length in mm by scale factor $\mu\ell$.

$$R_{14}^{\tau} = \frac{1}{l_{AC}} \cdot \left(F_{in4} \cdot h_4 + M_{in4}\right) = \frac{1}{0.05} \cdot \left(288 \cdot 0.014 + 25.6\right) = 516.03 \text{ N}.$$

As the magnitude of R_{14}^{τ} is positive we chose its direction correctly.

3. Determine the other unknown external forces that load Assur's group links (R_{05}, R_{14}^{n}, R_{14}).

Let us consider the equilibrium of the whole of Assur's group. As Assur's group is in a state of equilibrium the vector sum of all forces acting on the group links must be equal to zero ($\sum \overline{F_i} = 0$):

$$\underline{\underline{\overline{R_{14}^{\tau}}}} + \underline{\underline{\overline{F_{in4}}}} + \underline{\underline{\overline{F_5}}} + \underline{\underline{\overline{F_{in5}}}} + \underline{\overline{R_{65}}} + \underline{\overline{R_{14}^{n}}} = 0.$$

This vector equation does not contain forces $\overline{R_{54}}$ and $\overline{R_{45}}$ because their sum is equal to zero. Forces underlined by two lines are known both by direction and by magnitude. If a force is underlined by one line it is known by direction only.

The vector equation has two unknown parameters. We will solve this equation by plotting a force diagram. For that, we arbitrarily choose a scale factor for the force diagram and find the lengths of segments that represent the corresponding forces in the figure.

Let $\mu_F = 10\ \dfrac{N}{mm}$. Through pole *H* we lay off forces R^{τ}_{14}, F_{in4}, F_{5}, F_{in5} in succession marking vector ends by letters *a, b, c*, and *d,*

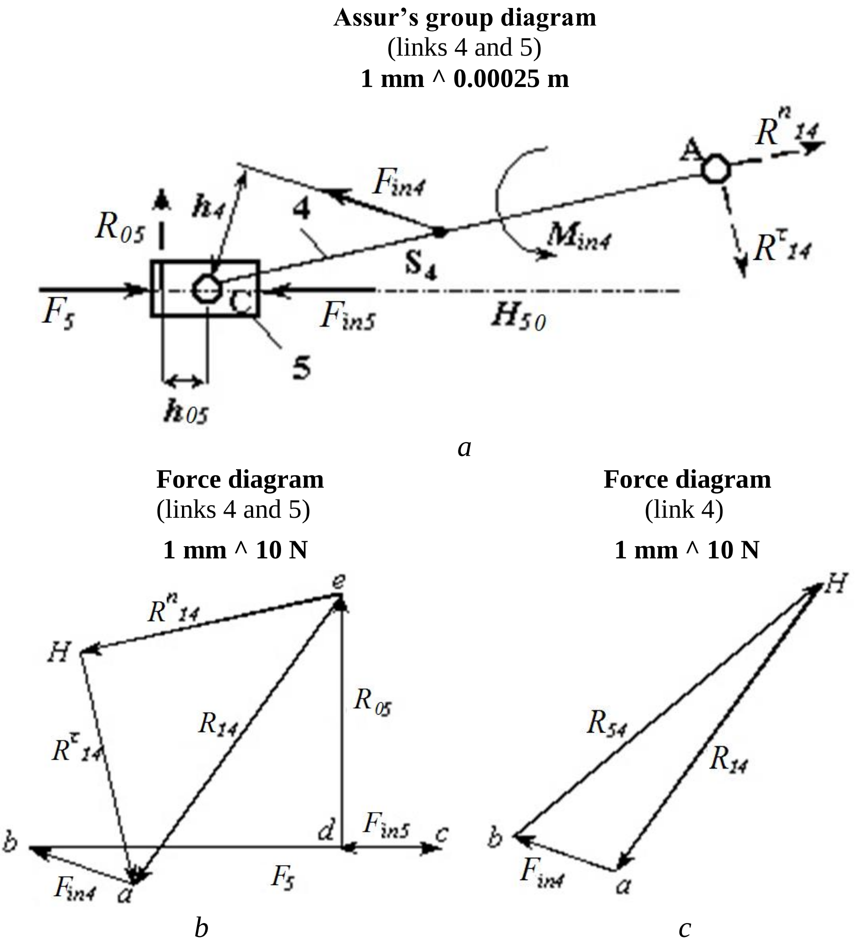


Fig. 5. Determination of forces in kinematic pairs of Assur's group formed by links 4 and 5: *a* – Assur's group diagram; *b* – force diagram for Assur's group; *c* – force diagram for link 4

correspondingly (Fig.5, *b*). The lengths of segments $\overline{Ha}$, $\overline{ab}$, $\overline{bc}$ and $\overline{cd}$ are determined in the following way:

$$\overline{Ha} = \frac{R_{14}^{\tau}}{\mu_F} = \frac{516.03}{10} = 51.6\,\text{mm},$$

$$\overline{ab} = \frac{F_{in4}}{\mu_F} = \frac{288}{10} = 28.8\,\text{mm},$$

$$\overline{bc} = \frac{F_5}{\mu_F} = \frac{1000}{10} = 100\,\text{mm},$$

$$\overline{cd} = \frac{F_{in5}}{\mu_F} = \frac{273}{10} = 27.3\,\text{mm}.$$

Through the obtained point *d* we draw a line parallel to R_{05} and through pole *H* we pass a line parallel to R_{14}^{n}. The point of intersection of these lines is point *e*. Segments $\overline{de}$, $\overline{aH}$ and $\overline{ea}$ represent corresponding forces R_{05}, R_{14}^{n}, R_{14} to scale. To determine the magnitudes of these forces, we should multiply corresponding segments by the scale factor.

$$R_{65} = \mu_F \cdot \overline{de} = 10 \cdot 56.5 = 565\ \text{N},$$

$$R_{14}^{n} = \mu_F \cdot \overline{eH} = 10 \cdot 62 = 620\ \text{N},$$

$$R_{14} = \mu_F \cdot \overline{ea} = 10 \cdot 80 = 800\ \text{N}.$$

The directions of these forces are determined according to the rule of vector composition (Fig.5, *b*). Thus, forces R_{05}, R_{14}^{n} and R_{14} were found.

4. Determine reacting forces in the internal kinematic pair of Assur's group (R_{54} and R_{45}).

For that, we should consider the state of equilibrium of either link 5 or link 4 and set up a vector equation of forces, acting on the link ($\sum \overline{F_i} = 0$). For example, let us consider the state of equilibrium of link 4:

$$\underline{\underline{\overline{R_{14}}}} + \underline{\underline{\overline{F_{in4}}}} + \overline{R_{54}} = 0.$$

This vector equation has two unknowns (the direction and the magnitude of force R_{54}). We solve this equation by plotting a force diagram. Taking into account the scale factor, we lay off R_{14}, F_{in4} in succession marking corresponding vector ends with letters *a* and *b*. After connecting point *b* with pole *H* we obtain segment $\overline{bH}$ that represents force R_{54} (Fig.5, *c*).

$$R_{54} = \mu_F \cdot \overline{bH} = 10 \cdot 93 = 930 \text{ N}.$$

According to the rule of vector composition we can determine the direction of force R_{54} (Fig.5, *c*).

5. Determine arm h_{05}, i.e. the point of application of force R_{05}.

For this purpose we will consider the state of equilibrium of link 5. The sum of moments of all acting forces with respect to point C must be equal to zero ($\sum M_C = 0$):

$$R_{05} \cdot h_{05} = 0.$$

Moments of forces R_{45}, F_5 and P_{in5} are equal to zero because their arms relative to point C are equal to zero too.

As force $R_{605} \neq 0$, we may make a conclusion that $h_{05} = 0$.

2.2.5.Determination of reacting forces in kinematic pairs of Assur's group formed by links 2 and 3

1. Plot Assur's group formed by links 2 and 3 at a given mechanism position taking into account the scale factor of length $\mu\ell = 0.00025 \ \frac{\text{m}}{\text{mm}}$ and apply all forces acting on the mechanism links (Fig.6, *a*). The action of separated links 1 and 6 is replaced by reacting forces R_{03}, R_{12} that develop in the corresponding kinematic pairs.

Link 3 is loaded by four forces such as pressure F_3, inertia force F_{in3}, force R_{03} that acts from the side of the fixed link 6 and force R_{23} that acts from the side of link 2. Link 2 is under the action of one moment of a couple of inertia forces M_{in2} and three forces: inertia force F_{in2}, force R_{12} from the side of link 1 and force R_{32} from the side of link 3. Unknown forces are: reacting force in the sliding kinematic pair R_{03}, forces $\overline{R_{23}} = -\overline{R_{32}}$ in internal turning kinematic pair B and force R_{12} that develops in external turning pair A.

Force R_{03} is perpendicular to guide of motion H_{30}, but the point of application of this force is unknown. That is why we mark the line of action of this force with a dotted line perpendicular to H_{50} that passes at certain distance h_{03} relative to point B (Fig.6, *a*). Arm h_{03} should be found.

Unknown force R_{12} is resolved into two components: normal force N_{12}^{n} that is parallel to AB and tangential force R_{12}^{τ} that is perpendicular to AB (Fig.6, *a*). The direction of R_{12}^{τ} is chosen arbitrarily.

2. Determine tangential component R_{12}^{τ} of the reacting force, that develops in turning pair A.

For this purpose we will consider the state of equilibrium of link 2. As this link is in a state of equilibrium, the sum of moments of all acting forces relative to any point must be equal to zero. Let us set up an equation of moments with respect to point B ($\sum M_B = 0$):

$$R_{12}^{\tau} \cdot l_{AB} + F_{in2} \cdot h_2 + M_{in2} = 0 .$$

Forces R_{12}^{n} and R_{32} do not form moments because their arms relative to point B are equal to zero. Arm h_2 is determined from the Assur's group diagram (Fig.6, *a*). For that, we should multiply the corresponding segment length in millimeters by scale factor $\mu\ell$.

$$R_{12}^{\tau} = -\frac{1}{l_{AB}} \cdot \left(F_{in2} \cdot h_2 + M_{in2}\right) = -\frac{1}{0.06} \cdot \left(280 \cdot 0.04 + 66.67\right) = -1297.83\ \text{N} .$$

Sign "-" shows that the direction of R_{12}^{τ} was chosen incorrectly. In this case it is necessary to change the direction of the reacting force in the Assur's group diagram (Fig.6, *a*).

Assur's group diagram
(links 2 and 3)
1 mm ^ 0.00025 m

Force diagram
(links 2 and 3)
1 mm ^ 20 N

Force diagram (link 2)
1 mm ^ 20 N

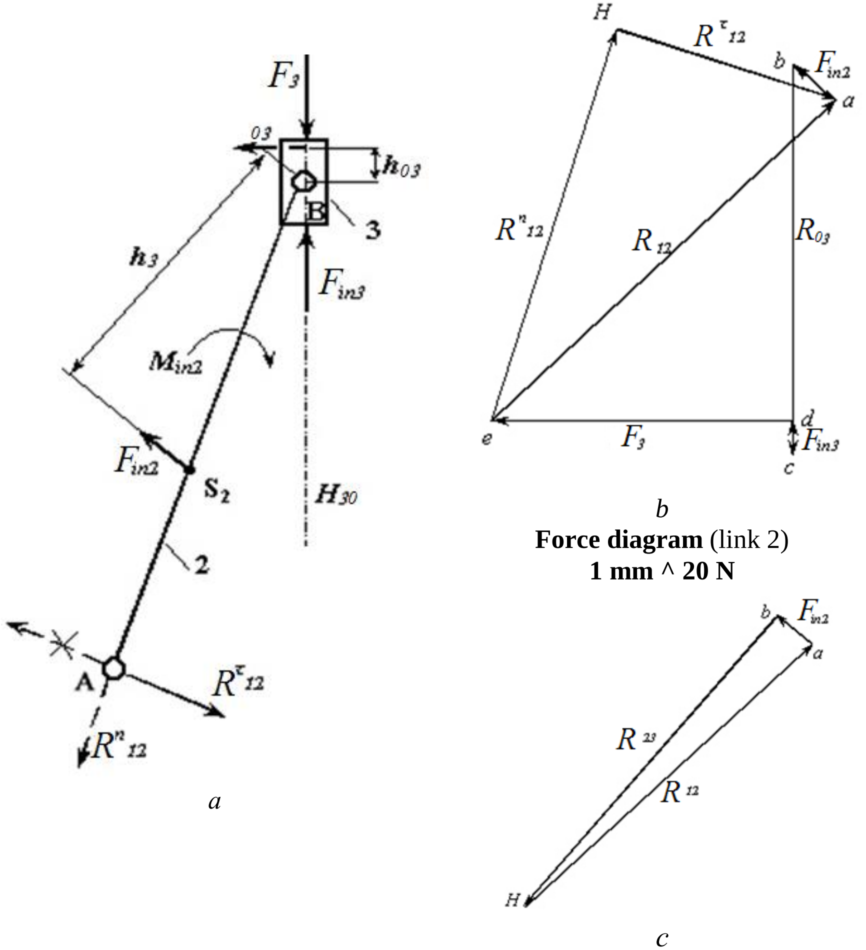


Fig. 6. Determination of forces in kinematic pairs of Assur's group formed by links 2 and 3: *a* – Assur's group diagram; *b* – force diagram for Assur's group; *c* – force diagram for link 2

3. Determine all unknown external forces that load the Assur's group links (R_{03}, R^{n}_{12}, R_{12}).

Let us consider equilibrium of the whole of Assur's group. As Assur's group is in a state of equilibrium the vector sum of all forces acting on the group links must be equal to zero ($\sum \overline{F_i} = 0$):

$$\underline{\underline{\overline{\mathrm{R}^{\tau}_{12}}}} + \underline{\underline{\overline{\mathrm{F}_{in2}}}} + \underline{\underline{\overline{\mathrm{F}_{3}}}} + \underline{\underline{\overline{\mathrm{F}_{in3}}}} + \underline{\overline{\mathrm{R}_{03}}} + \underline{\overline{\mathrm{R}^{n}_{12}}} = 0\,.$$

The vector equation has two unknown parameters. We will solve this equation by plotting a force diagram. For that, we arbitrarily choose the scale factor $\mu_F = 20\ \frac{\mathrm{N}}{\mathrm{mm}}$ of the force diagram and lay off forces R^{τ}_{12}, F_{in2}, F_3, F_{in3} in succession marking vector ends with letters *a*, *b*, *c*, and *d* correspondingly (Fig.6, *b*). Through obtained point *d* we draw a line parallel to R_{03} and through pole *H* we pass a line parallel to R^{n}_{12}. The point of intersection of these lines is marked as *e*. Segments $\overline{de}$, $\overline{aH}$ and $\overline{ea}$ represent corresponding forces R_{03}, R^{n}_{12}, R_{12} to scale. To determine the magnitudes of these forces, we should multiply corresponding segments by the scale factor.

$$\mathrm{R}_{03} = \mu_F \cdot \overline{de} = 20 \cdot 85.5 = 1710\ \mathrm{N}\,,$$

$$\mathrm{R}^{n}_{12} = \mu_F \cdot \overline{eH} = 20 \cdot 113 = 2260\ \mathrm{N}\,,$$

$$\mathrm{R}_{12} = \mu_F \cdot \overline{ea} = 20 \cdot 130 = 2600\ \mathrm{N}\,.$$

The direction of these forces is determined according to the rule of vector composition (Fig.6, *b*).

4. Determine reacting forces in internal kinematic pair B of Assur's group (R_{32} and R_{23}).

For that, we should consider the state of equilibrium of either link 2 or link 3 and set up a vector equation of forces, acting on the link ($\sum \overline{F_i} = 0$). For example, let us consider the state of equilibrium of link 2:

$$\underline{\underline{\overline{\mathrm{R}_{12}}}} + \underline{\underline{\overline{\mathrm{F}_{in2}}}} + \overline{\mathrm{R}_{32}} = 0\,.$$

This vector equation has two unknowns (the direction and the magnitude of force R_{32}). We solve this equation by plotting a force diagram. Taking into account the scale factor we lay off R_{12} , F_{in2} in succession marking corresponding vectors ends with letters *a* and *b*. After connecting point *b* with pole *H* we obtain segment $\overline{bH}$ that represents force R_{32} (Fig.6, *c*).

$$\mathrm{R}_{32} = \mu_{\mathrm{F}} \cdot \overline{bH} = 20 \cdot 125.5 = 2510 \text{ N}.$$

According to the rule of vector composition we can determine the direction of force R_{32} (Fig.6, *c*).

5. Determine arm h_{03} .

For this purpose we will consider the state of equilibrium of link 3. The sum of moments of all acting forces with respect to point B must be equal to zero ($\sum M_B = 0$):

$$R_{03} \cdot h_{03} = 0.$$

Moments of forces R_{23}, F_3 and F_{in3} are equal to zero because their arms relative to point B are equal to zero, too.

As force $R_{03} \neq 0$ consequently arm $h_{03} = 0$.

2.2.6. Force analysis of the group of initial links

1. Draw the group of initial links at the given position taking into account the scale factor $\mu\ell = 0.00025 \ \dfrac{\text{m}}{\text{mm}}$ and apply all forces that act on crank 1 (Fig.7, *a*).

The crank is under the action of one balancing moment M_{bal} that balances the action of all forces applied to the mechanism links and three forces: R_{41} from the side of link 4, R_{21} from the side of link 2 and R_{01} from the side of the fixed link 6.

Forces R_{41} and R_{21} are known. Their magnitudes are equal to corresponding forces R_{14} and R_{12} but the direction is opposite to the latter. Thus, in the force analysis of the group of initial links we should determine balancing moment M_{bal} and reacting force R_{01}.

2. Determine reacting force R_{01} that develops in turning kinematic pair O.

Let us consider equilibrium of crank 1. The vector sum of all forces acting on this link should be equal to zero ($\sum \overline{F_i} = 0$):

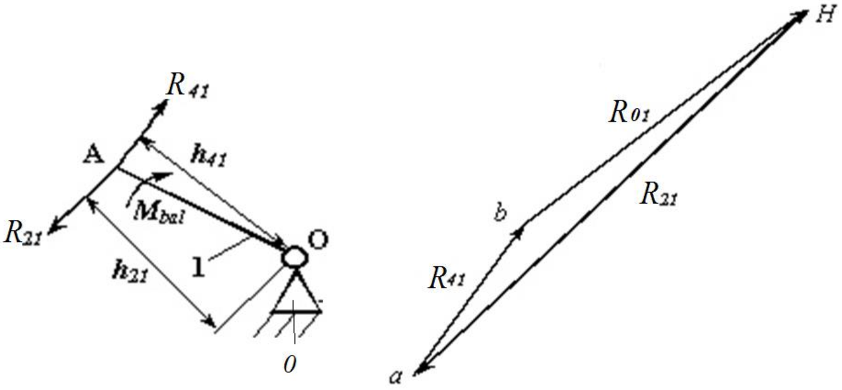
$$\underline{\underline{\overline{\mathrm{R}_{21}}}} + \underline{\underline{\overline{\mathrm{R}_{41}}}} + \overline{\mathrm{R}_{01}} = 0 .$$

We solve this equation by plotting a force diagram. After choosing the scale factor $\mu_F = 20 \ \dfrac{\mathrm{N}}{\mathrm{mm}}$, we lay off forces R_{41} and R_{21} marking vector ends with letters *a* and *b* (Fig.7, *b*). The obtained point *b* is connected with pole *H*. Segment $\overline{bH}$ represents reacting force R_{01} to scale. Let us determine the magnitude of this force:

$$\mathrm{R}_{01} = \mu_F \cdot \overline{bH} = 20 \cdot 88 = 1760 \ \mathrm{N} .$$

Diagram of the initial mechanism
1 mm ^ 0.00025 m

Force diagram
(link 1)
1 mm ^ 20 N



a *b*

Fig. 7. Determination of the balancing moment and reacting force N_{61} in kinematic pair O: *a* – diagram of the group of initial links; *b* - force diagram for link 1

3. Determine balancing moment M_{bal} .

For that, we set up an equation of moments of all forces relative to point O:

b

$$M_{bal} - R_{21} \cdot h_{21} + R_{41} \cdot h_{41} = 0 .$$

Moment of force R_{61} is equal to zero due to the fact that its arm with respect to point O is zero. Arms h_{21} and h_{41} are determined from the diagram of the group of initial links. For that, we multiply the corresponding segments in millimeters by $\mu\ell$.

$$\mathrm{M}_{bal} = \mathrm{R}_{21} \cdot h_{21} - \mathrm{R}_{41} \cdot h_{41} = 2600 \cdot 0.018 - 800 \cdot 0.019 = 31.6\,\mathrm{N} \cdot \mathrm{m} .$$

As the magnitude of M_{bal} is positive the direction was chosen correctly.

2.2.7. Determination of the balancing moment by Zhukovsky’s method

1. For the given mechanism position we plot the velocity diagram turned through 90° to an arbitrary scale. After that we will transfer all external forces that act on the mechanism links from the mechanism diagram to the corresponding points of the velocity diagram (Fig. 8).

The known moment M_{in2} is represented by a couple of forces F'_{in2} and F''_{in2} that are applied at points A and B perpendicular to AB (Fig. 4). The magnitude of these forces is determined by the formula

$$F'_{in2} = F''_{in2} = \frac{M_{in2}}{l_{AB}} = \frac{66.67}{0.06} = 1111.17\,\mathrm{N}.$$

In the same way we will represent moment M_{in4} by a couple of forces F'_{in4} and F''_{in4} that are applied at points A and C perpendicular to AC. The magnitude of these forces is determined as

$$F'_{in4} = F''_{in4} = \frac{M_{in4}}{l_{AC}} = \frac{25.6}{0.05} = 512\ N .$$

Unknown balancing moment M_{bal} is represented by a couple of forces F'_{bal} and F''_{bal} , that are applied at points A and O perpendicular to AO. The direction of these forces is chosen arbitrarily (Fig. 4).

2. Set up an equation of moments of all forces with respect to the pole of the velocity diagram and determine P'_{bal} and M_{bal}:

$$F'_{bal} \cdot \overline{pa} + F'_{in2} \cdot h'_2 - F'_{in4} \cdot h'_4 + F_3 \cdot \overline{pb} - F_{in3} \cdot \overline{pb} + F''_{in2} \cdot h''_2 +$$
$$+F_{in5} \cdot \overline{pc} - F_5 \cdot \overline{pc} - F''_{in4} \cdot h''_4 - F_{in2} \cdot h_2 + F_{in4} \cdot h_4 = 0.$$

Arms of all forces are substituted in the equation in millimeters. The lengths of these arms are determined from Fig. 8. As a result we obtain

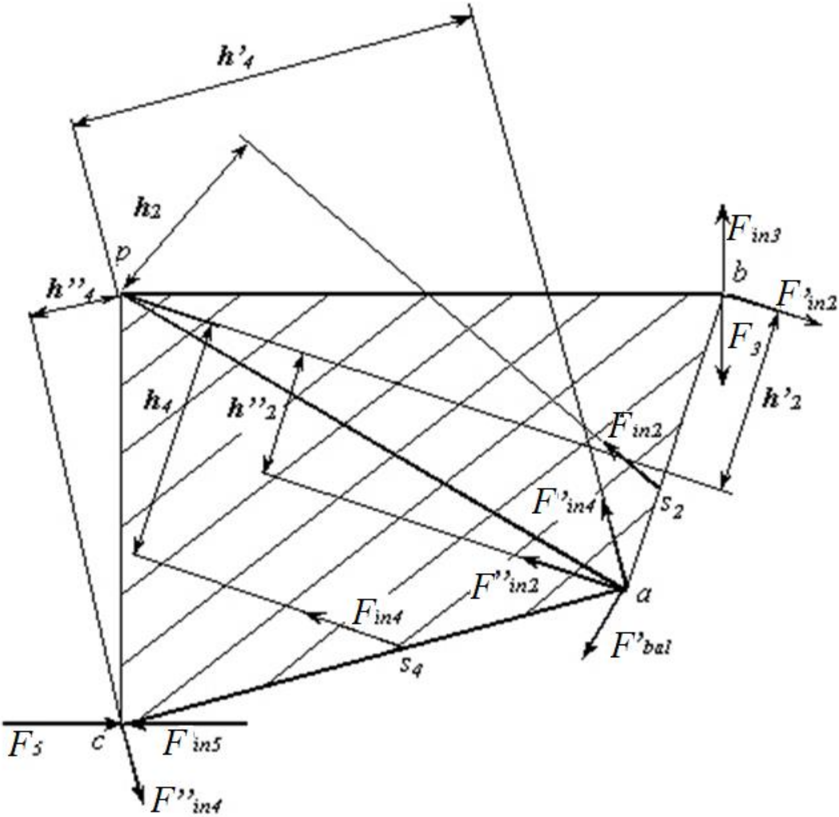


Fig. 8. Determining the balancing moment by Zhukovsky's method

$$F'_{bal} = \frac{1}{\overline{pa}}\left(-F'_{in2} \cdot h'_2 + F'_{in4} \cdot h'_4 - F_3 \cdot \overline{pb} + F_{in3} \cdot \overline{pb} - F''_{in2} \cdot h''_2 - F_{in5} \cdot \overline{pc} +\right.$$

$$\left.+F_5 \cdot \overline{pc} + F''_{in4} \cdot h''_4 + F_{in2} \cdot h_2 - F_{in4} \cdot h_4\right) = \frac{1}{100} \cdot \left(-1111.17 \cdot 12 +\right.$$

$$+512 \cdot 81 - 2000 \cdot 106 + 160 \cdot 106 - 1111.17 \cdot 36 - 273 \cdot 65 + 1000 \cdot 65 +$$

$$\left.+512 \cdot 12 + 280 \cdot 30 - 288 \cdot 40\right) = -1566.25\ N.$$

Sign “-“ shows that the direction of M_{bal} was chosen incorrectly. In reality the balancing moment is clockwise. The magnitude of this moment is determined as

$$M_{bal} = F'_{bal} \cdot l_{OA} = 1566.25 \cdot 0.02 = 31.325\ N \cdot m$$

3. Determine the difference between the balancing moments obtained by the two methods (combined static and inertia force analysis and Zhukovsky’s method):

$$\Delta M_{bal} = \frac{31.6 - 31.325}{31.325} \cdot 100\% = 0.88\,\%.$$

The error must be not more than 5 %.

Recommended literature

1. *Баранов Г.Г.* Курс теории механизмов и машин. – М.: Машиностроение, 1975. – 496 с.

2. *М.Ф.Воронкін, А.А.Цимбалюк.* Основи теорії механізмів і машин: Конспект лекцій. - К.: КМУЦА, 2000. –208 с.

3. *Кинематическое* исследование рычажных механизмов: Методические указания к курсовой работе.– К.: КИИГА, 1989, - 40 с.

4. *Теория* механизмов и машин. Силовое исследование рычажных механизмов: Методические указания к курсовой работе. –К.: КИИГА, 1989. – 38 с.

PREFACE

The term paper on the subject *Theory of Mechanisms and Machines* is one of the basic kinds of the student's individual work. The purpose of the term paper is to enhance the knowledge acquired by the student in the lectures, practical classes and laboratory sessions, and develop the skills of making research and design of present-day aircraft mechanisms and machines.

The term paper is to include the following parts:

1. Kinematic and force analyses of a leverage.
2. Designing a planetary gearing.
3. Designing an involute gearing.

Each part of the term paper should consist of a calculation and graphical sections.

All calculation sections are to be presented as an explanatory note that should be carried out according to requirements of «ДСТУ 3008-95. Державний стандарт України. Документація. Звіти в сфері науки і техніки. Структура і правила оформлення». The explanatory note is either typed or hand written in blue or black ink on one side of size A4 paper. Every sheet is to be paginated and have the following margins: top – 5 mm, bottom – 5 mm, right – 5 mm, and left – 20 mm.

Besides calculation sections an explanatory note should have the contents table, the assignment, the list of literature used in working on the term paper. Each new part is to begin with a new page.

Each part must be subdivided into items marked with numerals separated by a point. The first numeral represents the number of the part, the second one shows the number of the item.

Calculations should be made in an order that corresponds to the graphical plots. All magnitudes that are part of formulas should be explained. In addition, it is necessary to denote units of measurement of parameters calculated.

The graphical section is to be executed on size A1 whatman paper (part1) and A2 (part 2 and part 3) in pencil. Above every drawing there should be an inscription indicating the scale. The title block should be drawn in the bottom right hand corner.

1. DESIGNING PLANETARY GEAR TRAINS

1.1 Theoretical information

The planetary gear train is a mechanism in which geometrical axes of one or several gears can move relative to the frame.

Planetary gear trains are divided into four groups:

- differential gear mechanisms that have two or more degrees of freedoms;
- planetary gearings having one degree of freedom;
- closed differential gear mechanisms obtained from ordinary differential gear mechanisms by constraining two main links with a simple gearing;
- combined gear trains consisting of planetary gearings and simple gearings joined with each other in succession.

In comparison with the other gear trains planetary gear trains have the following advantages:

- small size and mass. This follows from the fact that the power is transmitted through several routes at the same time, the number of routes being equal to the number of planet pinions. Accordingly, the load imposed on the teeth in the meshing gears falls to a fraction of its original value;
- the input and output shafts are arranged coaxially simplifying the layout of machines;
- planetary gear trains are less noisy in operation than ordinary gear trains, because toothed wheels of the planetary gear train are smaller and the forces balance out one another when the planets are arranged symmetrically.
- lighter loads on the bearings result in minimum loses and simpler designs;
- they have high velocity ratios with small overall dimensions;
- they allow to compose and decompose motions. For example, transmission of motion from two independent motors to one driven link or from one motor to two driven links by means of differential gear mechanisms.

On the other hand, planetary gear trains require high accuracy of manufacturing and assembling, their efficiency falls with rising the velocity ratio.

Planetary gear trains may be used as speed reducers with a constant velocity ratio in power transmissions and various devices, gearboxes where the velocity ratio can be varied by locking appropriate members, and differential gear mechanisms of automobiles, tractors, machine tools and like that.

Design (synthesis) of a planetary gear train consists of two stages:

- selection of the mechanism diagram taking into accounts its purpose and efficiency; and
- determination of a number of teeth of all toothed wheels to provide the given velocity ratio.

As a rule, planetary gear trains are formed by means of standard involute straight spur gears.

Selection of the planetary gear train diagram

Fig.1.1 shows four diagrams of planetary gearings that have found the most wide application in the mechanical engineering.

Every of planetary gearings has one degree of freedom and consists of four links. Toothed wheels z_1 and z_3 whose geometrical axes coincide with the main axis of the mechanism are called *sun gears*. Toothed wheel z_2 having a movable axis is named a *planet pinion*. The shaft of the planet pinion rotates in the bearing B, which is mounted on link 4, named a *driver* or *carrier*, and together with this link, rotates around the main axis of the planetary gear train. The planetary gearing may contain single (Fig.1.1, *a*) or double planet pinions (Fig.1.1, *b, c, d*).

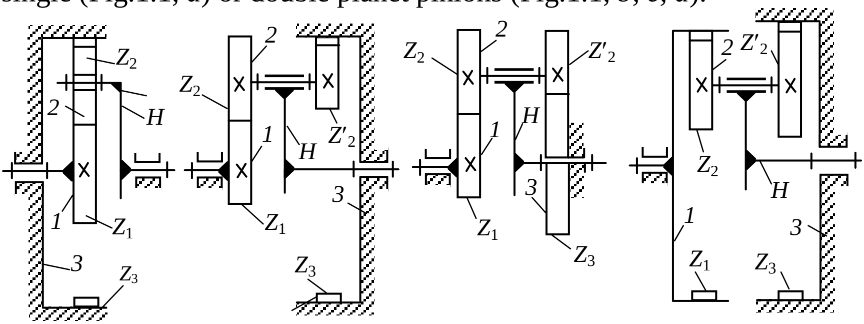


Fig.1.1. Typical diagrams of planetary gearing

If in a planetary gear train the immovable sun gear becomes movable, we deal with a differential gear mechanism that consists of five links and has two degrees of freedom.

In order to carry out kinematic analysis of the planetary gear train we should use the method of reversed motion. According to this method it is necessary to add the rotational speed of the driver with opposite sign to rotational speeds of all mechanism links. In this case rotational speeds of links 1, 2, 3 and 4 are correspondingly equal to (n_1 - n_H), (n_2 - n_H), (n_3 - n_H) and (n_4- n_H) = 0. Thus, driver H becomes immovable and a planetary gear train is transformed into ordinary gear train with fixed axes of toothed wheels. Diagrams of reversed mechanisms for mentioned above planetary gearings (Fig.1.1) are shown in Fig.1.2.

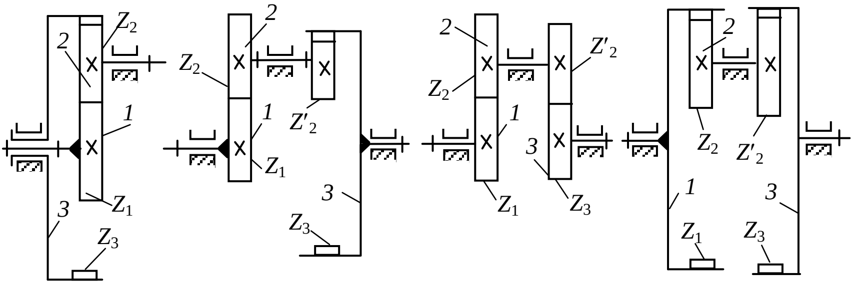


Fig.1.2. Diagrams of reversed mechanisms

The velocity ratio of the reversed mechanism from link 1 to link 3 u_{13}^H may be found as product of velocity ratios of single stage gearings that are a part of the mechanism:

$$u_{13}^H = u_{12}^H \cdot u_{23}^H .$$

For diagrams shown in Fig.2 u'_{13} is calculated in the following way:

$$u_{13}^H = u_{12}^H u_{23}^H = \left(-\frac{z_2}{z_1}\right)\left(+\frac{z_3}{z_2}\right) = -\frac{z_3}{z_1} \qquad \text{for Fig 1.2, } a,$$

$$u_{13}^H = u_{12}^H u_{23}^H = \left(-\frac{z_2}{z_1}\right)\left(+\frac{z_3}{z_2'}\right) = -\frac{z_2 z_3}{z_1 z_2'} \qquad \text{for Fig 1.2, } b,$$

$$u_{13}^H = u_{12}^H u_{23}^H = \left(-\frac{z_2}{z_1}\right)\left(-\frac{z_3}{z_2'}\right) = \frac{z_2 z_3}{z_1 z_2'} \qquad \text{for Fig 1.2, } c,$$

$$u_{13}^H = u_{12}^H u_{23}^H = \left(+\frac{z_2}{z_1}\right)\left(+\frac{z_3}{z_2'}\right) = \frac{z_2 z_3}{z_1 z_2'} \qquad \text{for Fig 1.2, } d,$$

where z_1, z_2, z'_2 and z_3 are number of teeth of corresponding toothed wheels.

On the other hand, the velocity ratio u'_{13} may be determined as ratio of rotational speeds of links 1 and 3. Taking into account that in the reversed mechanism the rotational speed of link 1 is (n_1-n_H) and the rotational speed of link 3 is (n_3-n_H) = -n_H we obtain:

$$u_{13}^{(H)} = \frac{n_1 - n_H}{n_3 - n_H} = \frac{n_1 - n_H}{-n_H} = -\frac{n_1}{n_H} + 1 = -u_{1H} + 1,$$

where u_{14} is the velocity ratio of the planetary gearing.

Then

$$u_{1H} = 1 - u_{13}^{H}.$$

The obtained formula allows to determine the velocity ratio of the planetary gearing. For this purpose it is necessary to reverse a mechanism and to find its velocity ratio.

The simplest planetary gearings may be divided into two groups:

- planetary gearings with positive value of the velocity ratio of the reversed mechanism (Fig.1.1, *c, d*);
- planetary gearings with negative value of the velocity ratio of the reversed mechanism (Fig.1.1, *a, b*).

Mechanisms of the first group ($u_{13}^{(H)}$>0) are formed by gears with either only external (Fig.1.2, *c*) or only internal (Fig.1.2, *d*) toothing. As a rule, in these mechanisms the driving link is the driver. Planetary gearings of this group are characterized by the high velocity ratio. For example, if in the mechanism with two gears of external contact (Fig.1.1, *c*) we assume z_1 = z'_2 =100, z_2 = 99, z_3 = 101 the velocity ratio from driver 4 to sun gear 1 (u_{41}) will be 10 000. But in this case the efficiency is less than 1 %. That is why planetary gearings of this group are used in non-power short-term transmissions. The most rational values of the velocity ratio u_{41} of mentioned above mechanisms are ranged from 30 to 1700.

Mechanisms of the second group ($u_{13}^{(H)}$<0) consist of gears with both external and internal contact. They may be with either single (Fig.1.1, *a*) or double (Fig.1.1, *b*) planet pinion. Mechanisms of this group are used in power and auxiliary drives as multi-planet pinions speed reducers of medium and high power with the velocity ratio u_{14}

ranged from 2 to 15 and the efficiency from 96 to 99 %. Planetary gearings with a single planet pinion (Fig.1.1, *a*) have the highest efficiency (97-99 %), small overall dimensions in the axial direction. Besides, they are the most compact. That is why they find very wide application in the mechanical engineering. We may meet these mechanisms in remote control plants, in aircraft drives, Moon research vehicles and others.

In order to obtain the high velocity ratio multi-stage planetary gearings are used that are formed as a result of successive connection of simple planetary gearings. The example of this mechanism is shown in Fig.1.3. It consists of three planetary gearings with single planet pinion.

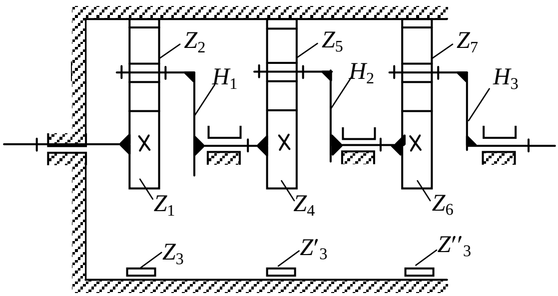


Fig. 1.3. Three-stage planetary gear train

The velocity ratio of the mentioned above mechanism is determined as product of velocity ratios of simple planetary gearings

$$u_{1H} = u_{1H_1} u_{4H_2} u_{6H_3} \, ,$$

where

$$u_{1H} = 1 - u_{13}^{H_1} = 1 + \frac{z_3}{z_1}; \quad u_{4H_2} = 1 - u_{43}^{H_2} = 1 + \frac{z_3'}{z_4}; \quad u_{6H_3} = 1 - u_{63}^{H_3} = 1 + \frac{z_3''}{z_6}.$$

If the velocity ratio of every stage is 5 ($u_{1H_1} = u_{4H_2} = u_{6H_3} = 5$), the total velocity ratio of the whole mechanism is u_{1H3} = 125 with relatively high efficiency (88 – 94 %). The overall dimensions of this triple-stage planetary gear speed reducer are less in comparison with the speed reducer with fixed axes of gears (when the power and the velocity ratio are identical).

Determination of a number of teeth of planetary gearing toothed wheels

After choosing the planetary gear train diagram it is necessary to determine a number of teeth of mechanism toothed wheels. In this case the following conditions should be carried out: coaxiality condition, mating condition, coincidence condition and condition of the right engagement.

Coaxiality condition

In order to provide the engagement of planet pinions with sun gears both sun gears and the driver must have the common geometrical axis of rotation. In this case, the centre distance between the movable sun gear and the planet pinion should be equal to the centre distance between the immovable sun gear and the planet pinion. For planetary gear trains shown in Fig.1.1 the condition of coaxiality has the following form:

$r_1 + r_2 = r_3 - r_2$ or $z_1 + z_2 = z_3 - z_2$ for Fig. 1.1, *a*,

$r_1 + r_2 = r_3 - r_2'$ for Fig.1.1, *b*,

$r_1 + r_2 = r_3 + r_2'$ for Fig.1.1, *c*,

$r_1 - r_2 = r_3 - r_2'$ for Fig.1.1, *d*,

where r_1 and r_3 are nominal pitch circle radii of the movable and immovable sun gears correspondingly; r_2 and r_2' are nominal pitch circle radii of the planet pinions.

Mating condition

This condition takes into account necessity of collocation of several planet pinions along a circle in one plane. According to this condition the addendum circles of mating planet pinions should not be intersected or touch each other. The mating condition is carried out when

$$O_2O_2' > 2 \cdot r_{a2}\ , \qquad (1.1)$$

where O_2O_2' is the centre distance between two adjacent planet pinions (Fig.1.4); $r_{a2} = 0.5 \cdot m \cdot (z_2+2)$ is the addendum circle diameter of the planet pinion.

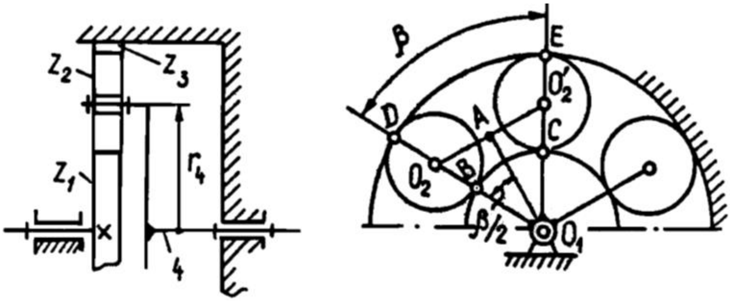


Fig.1.4. For determination of the mating and coincidence conditions

The centre distance $O_2O_2^{'}$ between two adjacent planet pinions is determined as

$$O_2O_2^{'} = 2 \cdot O_2A = 2 \cdot O_1O_2 \cdot \sin\left(\frac{\beta}{2}\right), \qquad (1.2)$$

where $\beta = \frac{2\pi}{k}$ (k is a number of planet pinions).

The centre distance O_1O_2 between the planet pinion and the movable sun gear is

$$O_1O_2 = r_1 + r_2 = 0.5 \cdot m \cdot (z_1 + z_2). \qquad (1.3)$$

After substituting (1.3) to (1.2) we obtain

$$O_2O_2^{'} = m \cdot (z_1 + z_2) \cdot \sin\left(\frac{\pi}{k}\right).$$

Then the mating condition (1.1) has the following form:

$$(z_1 + z_2)\sin\left(\frac{\pi}{k}\right) > (z_2 + 2),$$

or

$$\sin\left(\frac{\pi}{k}\right) > \frac{z_2 + 2}{z_1 + z_2}.$$

Coincidence condition

This condition takes into account necessity of simultaneous engagement of all planet pinions with both sun gears when angles between planet pinions β is identical.

After installing the first planet pinion the movable sun gear has certain position. If we do not carry out certain requirements after installing the other planet pinions tops of their teeth may not coincide with spaces of one of sun gears. In this case, mechanism assembling is impossible.

Coincidence condition is carried out when the curve formed by parts BC and DE of pitch circles of both sun gears (Fig.1.4) consists of the whole number of pitches, that is (i.e.) when

$$BC+DE=\gamma,$$

where γ is any integer numeral.

Let us assume that arc BC consists of the whole number of pitches γ_1 and remainder s_1:

$$BC=\gamma_1\cdot p+s_1 .$$

In the same way, we may write that

$$DE=\gamma_2\cdot p+s_2 .$$

Arcs BC and DE can also be determined by sun gears number of teeth:

$$BC=\frac{p\cdot z_1}{k},\ DE=\frac{p\cdot z_3}{k},$$

where z_1 and z_3 are number of teeth of movable and immovable sun gears correspondingly; k is number of planet pinions.

The sum of these arcs is found as

$$BC+DE=\frac{p}{k}\cdot(z_1+z_3)=p\cdot(\gamma_1+\gamma_2)+(s_1+s_2),$$

whence

$$(z_1+z_3)=k\cdot(\gamma_1+\gamma_2)+\frac{k\cdot(s_1+s_2)}{p}. \qquad (1.4)$$

As (z_1+z_3) is an integer numeral then the right part of (1.4) has to be an integer numeral too. It is possible when $(s_1+s_2) = p$. After substituting $(s_1+s_2) = p$ to (1.4) we obtain

$$(z_1+z_3)=k\cdot(\gamma_1+\gamma_2+1)=k\cdot\gamma,$$

or

$$\frac{(z_1+z_3)}{k}=\gamma,$$

where γ is an integer numeral.

Thus, coincidence condition is carried out when the sum of sun gears number of teeth is devisable by the number of planet pinions k.

Condition of right engagement

This condition takes into account absence of teeth undercutting and interference. In order to eliminate these phenomena a number of teeth of standard involute spur gears with external toothing (when α = $20°, h_a^* = 1$) should not be less than 17. For gears with internal toothing a number of teeth of the annual gear has to be greater or equal to 85 and the pinion must have not less than 20 teeth. In this case, the difference between number of teeth of the annular gear and the pinion has to be not less than 8.

In order to determine number of teeth z_1, z_2 and z_3 of toothed wheels of the planetary gear train with a single planet pinion (Fig.1.1, *a*) we should set up three equations. For planetary gear trains with a double planet pinion (Fig.1.1, *b, c, d*) it is necessary to write four equations to find z_1, z_2, z_2' and z_3. However, it is possible to set up two equations only: the first equation is determination of the velocity ratio and the second one is the equation of coaxiality. Thus, solution of this problem is multivariant.

1.2. Examples of synthesis of planetary gear trains

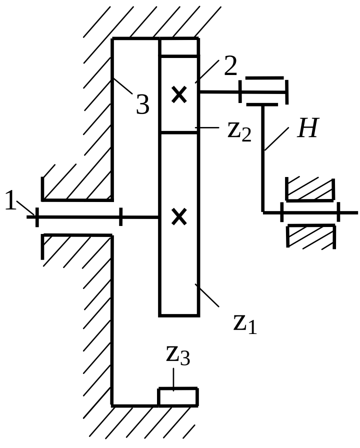


Fig.1.5. Planetary gearing with a single planet pinion

Example 1

Determine number of teeth of gears and carry out kinematic analysis of the planetary gear speed reducer with a single planet pinion shown in Fig.1.5 if number of planet pinions k = 4; rotational speed of the input shaft n_1 = 5600 rpm; rotational speed of the output shaft n_H = 1400 rpm; module of gears *m* = 3 mm. Number of teeth of gears must satisfy to the following condition $17 \le z \le 180$. Gears have to be manufactured by standard cutter with $\alpha = 20^o$ and $h_a^* = 1$.

Solution

1. Determination of a number of teeth of the planetary gearing toothed wheels.

1.1. Determine the velocity ratio of the planetary gearing:

$$u_{14} = 1 - u_{13}^{\text{í}} = 1 + \frac{z_3}{z_1} = \frac{n_1}{n_{\text{í}}} = \frac{5600}{1400} = 4,$$

where $u_{13}^{\text{í}} = -\frac{z_3}{z_1}$ is the velocity ratio of the reversed mechanism.

1.2. Write the main conditions for the planetary gearing:

- coaxiality condition is

$$z_1 + z_2 = z_3 - z_2;$$

- mating condition is

$$\sin\left(\frac{\pi}{k}\right) > \frac{z_2 + 2}{z_1 + z_2};$$

- coincidence condition is

$$\frac{z_1 \cdot u_{14}}{k} = \gamma,$$

where γ is any integer numeral;

- condition of right engagement (when $\alpha = 20^{\circ}$ and $h_a^* = 1$) is

$$z_1 \geq 17,\ \ z_2 \geq 20,\ \ z_3 \geq 85,\ \ z_3 - z_2 \geq 8.$$

1.3. Set up equations for determination of gears number of teeth:

$$u_{1\text{í}} = 1 + \frac{z_3}{z_1}, \qquad z_1 + z_2 = z_3 - z_2, \qquad \frac{z_1 \cdot u_{1\text{í}}}{k} = \gamma.$$

Let us find z_2 and z_3 taking into account mentioned above equations

$$z_3 = z_1 \cdot (u_{1\text{í}} - 1);$$

$$z_2 = \frac{z_3 - z_1}{2} = \frac{z_1 \cdot (u_{1\text{í}} - 2)}{2}.$$

In order to determine gears number of teeth we will set up the following system of relations

$$z_1 : z_2 : z_3 : \gamma = z_1 : \frac{z_1\left(u_{1H} - 2\right)}{2} : z_1\left(u_{1H} - 1\right) : \frac{z_1 u_{1H}}{k}$$

The obtained system is considered as the basic equation for determination of the gears number of teeth of the planetary gearing. It provides carrying out the coaxiality condition and the coincidence condition when the velocity ratio of the planetary gearing is known. That is why after choosing gears number of teeth it is enough to make checking for the mating condition and the condition of right engagement.

1.4. Choose number of teeth z_1 arbitrarily as less as possible. According to the condition of right engagement and the coincidence condition it should be greater or equal to 17 and divisible by k. Let $z_1 = 20$.

1.5. Determine gears number of teeth of the planetary gearing:

$$z_1 : z_2 : z_3 : \gamma = 20 : \frac{20 \cdot (4-2)}{2} : 20 \cdot (4-1) : \frac{20 \cdot 4}{4},$$

$$z_1 : z_2 : z_3 : \gamma = 20 : 20 : 60 : 20.$$

Consequently

$$z_1 = 20, \; z_2 = 20, \; z_3 = 60.$$

1.6. Check the mating condition:

$$\sin\left(\frac{\pi}{k}\right) > \frac{z_2 + 2}{z_1 + z_2},$$

$$\sin\left(\frac{\pi}{4}\right) = \sin\left(\frac{180}{4}\right) = \sin 45° = 0.7071,$$

$$\frac{z_2 + 2}{z_1 + z_2} = \frac{20 + 2}{20 + 20} = 0.55.$$

The mating condition is carried out because $\sin 45^{\circ} > 0.55$.

1.7. Check the condition of right engagement:

$$z_1 \geq 17, \; z_2 \geq 20, \; z_3 \geq 85, \; z_3 - z_2 \geq 8.$$

In our case this condition is not carried out because

$$z_1 > 17, \; z_2 \geq 20, \; z_3 < 85, \; z_3 - z_2 \geq 8.$$

1.8. Reselect number of teeth z_1 and recalculate the number of teeth of all gears.

Let $z_1 = 30$. Then

$$z_1 : z_2 : z_3 : \gamma = 30 : \frac{30 \cdot (4-2)}{2} : 30 \cdot (4-1) : \frac{30 \cdot 4}{4},$$

$$z_1 : z_2 : z_3 : \gamma = 30 : 30 : 90 : 30.$$

Thus

$$z_1 = 30, \;\; z_2 = 30, \;\; z_3 = 90.$$

1.9. Check the mating condition:

$$\sin\left(\frac{\pi}{k}\right) > \frac{z_2 + 2}{z_1 + z_2},$$

$$\sin\left(\frac{\pi}{4}\right) = \sin\left(\frac{180}{4}\right) = \sin 45^\circ = 0.7071,$$

$$\frac{z_2 + 2}{z_1 + z_2} = \frac{30 + 2}{30 + 30} = 0.533.$$

The mating condition is carried out because $\sin 45^\circ > 0.533$.

1.10. Check the condition of right engagement:

$$z_1 \geq 17, \;\; z_2 \geq 20, \;\; z_3 \geq 85, \;\; z_3 - z_2 \geq 8.$$

In our case

$$z_1 = 30 > 17, \;\; z_2 = 30 > 20, \;\; z_3 = 90 > 85, \;\; z_3 - z_2 = 60 > 8.$$

Thus, the condition of right engagement is carried out.

1.11. Check the velocity ratio

$$u_{1í} = 1 - u_{13}^{í} = 1 + \frac{z_3}{z_1} = 1 + \frac{90}{30} = 4.$$

Thus, gears number of teeth of the planetary gearing (z_1 = 30, z_2 = 30, z_3 = 90) was determined correctly. They provide given velocity ratio of the mechanism and fulfilling all necessary conditions.

2. Determination of gears diameters:

$$d_{w1} = d_1 = m \cdot z_1 = 3 \cdot 30 = 90 \text{ mm},$$

$$d_{w2} = d_2 = m \cdot z_2 = 3 \cdot 30 = 90 \text{ mm},$$

$$d_{w3} = d_3 = m \cdot z_3 = 3 \cdot 90 = 270 \text{ mm}.$$

3. Kinematic analysis of the planetary gearing.

3.1. Plot the mechanism diagram (Fig.1.6, *a*). For that we choose arbitrarily the length of a segment that represents the largest gear diameter. In our case it is the annular gear diameter d_3. The length of this segment has to be greater or equal to 220 mm. Let $\overline{d_3}$ = 270 mm. Then the scale factor is

$$\mu_l = \frac{d_3}{\overline{\overline{d_3}}} = \frac{0.270}{270} = 0.001\,\frac{m}{mm}.$$

3.2. Plot the velocity diagram of the mechanism.

3.2.1. Determine the linear velocity of point P_1 located on sun gear 1:

$$V_{P1} = \frac{\pi \cdot n_1}{30} \cdot r_1 = \frac{3{,}14 \cdot 5600 \cdot 0{,}045}{30} = 26.376\,\frac{m}{sec}.$$

3.2.2. Choose the length of segment $\overline{P_1'a}$ that represents V_{P1} on the velocity diagram (Fig.1.6, *b*). The length of this segment should not be less than 150 mm. Let $\overline{P_1'a}$ =168 mm. Then the scale factor of the velocity diagram is

$$\mu_V = \frac{V_{P1}}{\overline{\overline{P_1'a}}} = \frac{26.376}{168} = 0.157\,\frac{m/sec}{mm}.$$

3.2.3. Draw straight line MM perpendicular to the mechanism common axis and transfer points O_1, O_2, P_1 and P_2 to this line. As a result we will have points O_1', O_2', P_1' and P_2'.

3.2.4. Plot the velocity diagram for sun gear 1. For this purpose we lay off segment $\overline{P_1'a}$ perpendicular to line MM and connect obtained point *a* with point O_1'. Segment $\overline{O_1'a}$ shows the distribution law of velocities of the movable sun gear points.

3.2.5. Plot the velocity diagram for planet pinion 2. Velocities of two points (P_1 and P_2) of this gear have been known. At point P_1 the velocity of the planet pinion is equal to the velocity of the movable sun gear V_{P1}. At point P_2 the planet pinion velocity is zero because point P_2 is the pitch point of planet pinion 2 and immovable sun gear 3. After connecting points P_2' and *a* we obtain the velocity diagram for the planet pinion. The velocity of the planet pinion center O_2 will be represented by segment $\overline{O_1'b}$ and is determined as

$$V_{O2} = \mu_V \cdot \overline{O_1'b} = 0.157 \cdot 84 = 13.188 \text{ m/sec}.$$

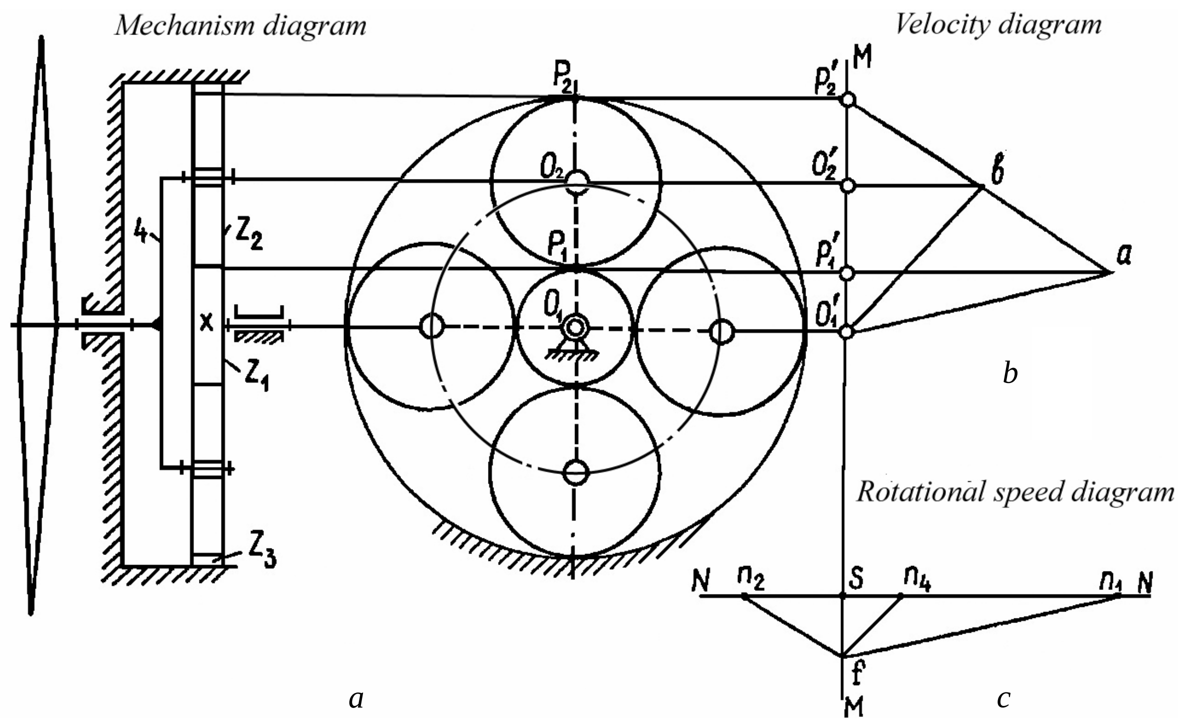


Fig.1.6. Planetary gear speed reducer: *a* – mechanism diagram; *b* – velocity diagram; *c* - rotational speed diagram

3.2.6. Plot the velocity diagram for driver H. The velocity of point O_2 of the planet pinion is simultaneously the velocity of the bearing center of the driver that rotates around point O_1. After connecting point $O_1^{'}$ with point b we obtain segment $\overline{O_1^{'}b}$ that is the distribution law of velocities of driver points.

3.3. Plot the rotational speed diagram.

3.3.1. Draw straight line NN parallel to the mechanism axis through arbitrarily chosen point s located on line MM (Fig.1.6, c). Then we lay off segment $\overline{sf}$ of arbitrary length along line MM. The length of $\overline{sf}$ should be greater or equal to a segment that represents the radius of the smaller sun gear pitch circle. Through point f we draw straight lines parallel to segments $\overline{O_1{}'a}$, $\overline{O_1'b}$ and $\overline{P_2'a}$ of the velocity diagram. Intersections of these lines with line NN are marked by points n_1, n_H and n_2 correspondingly. Segments $\overline{sn_1}$, $\overline{sn_2}$ and $\overline{sn_4}$ will represent rotational speeds of sun gear 1, planet pinion 2 and driver H to certain scale.

Let $\overline{sf}$ = 50 mm. Then the scale factor of the rotational speed diagram is

$$\mu_n = \frac{30 \cdot \mu_V}{\pi \cdot \overline{sf} \cdot \mu_l} = \frac{30 \cdot 0.157}{3.14 \cdot 50 \cdot 0.001} = 30\,\frac{\text{rpm}}{\text{mm}}.$$

3.3.2. Determine rotational speeds of planetary gearing links:

$$n_1 = \mu_n \cdot \overline{sn_1} = 30 \cdot 187 = 5610 \text{ rpm},$$

$$n_2 = \mu_n \cdot \overline{sn_2} = 30 \cdot 94 = 2820 \text{ rpm},$$

$$n_H = \mu_n \cdot \overline{sn_4} = 30 \cdot 47 = 1410 \text{ rpm}.$$

3.3.3. Determine the velocity ratio of the planetary gearing:

$$u_{1H} = \frac{n_1}{n_H} = \frac{\mu_n \cdot \overline{sn_1}}{\mu_n \cdot \overline{sn_4}} = \frac{\overline{sn_1}}{\overline{sn_4}} = \frac{187}{47} = 3.98.$$

The difference between the planetary gearing velocity ratio obtained by the analytical and graphical methods has to be less than 4 %. In our case

$$\Delta u_{1H} = \frac{4 - 3.98}{4} \cdot 100\,\% = 0.5\,\%.$$

Example 2

Determine the number of teeth of toothed wheels and carry out kinematic analysis for the planetary gear speed reducer shown in Fig.1.7 if number of planet pinions k = 3; rotational speed of the input shaft n_1 = 8400 rpm; rotational speed of the output shaft n_H =600 rpm; module of planetary gearing gears m = 3 mm. Number of teeth of gears must satisfy to the following condition $17 \leq z \leq 180$. Toothed wheels have to be manufactured by standard cutter with $\alpha = 20^o$ and $h_a^* = 1$.

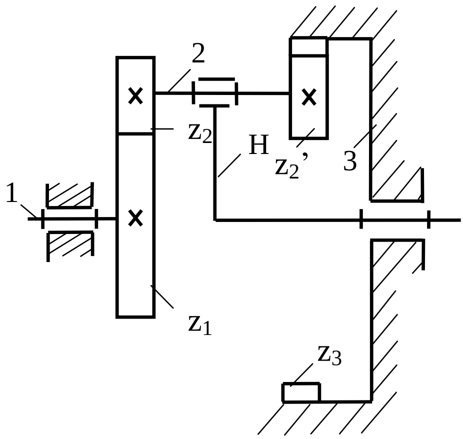


Fig.1.7. Planetary gearing with double planet pinions

Solution

1. Determination of the number of teeth of the planetary gearing toothed wheels.

1.1. Determine the velocity ratio of the planetary gearing

$$u_{1H} = 1 - u_{13}^{H} = 1 + \frac{z_2 z_3}{z_1 z_2'} = \frac{n_1}{n_H} = \frac{8400}{600} = 14,$$

where $u_{13}^{\acute{I}} = -\dfrac{z_2 z_3}{z_1 z_2'}$ is the velocity ratio of the reversed mechanism.

1.2. Write the main conditions for the planetary gearing:

- coaxiality condition is

$$m_{12} \cdot (z_1 + z_2) = m_{2'3} \cdot (z_3 - z_2'),$$

where m_{12} is a module of gears z_1 and z_2; $m_{2'3}$ is a module of gears z_2' and z_3. This condition may be written in the following way

$$q \cdot (z_1 + z_2) = (z_3 - z_2'),$$

where $q = \dfrac{m_{12}}{m_{2'3}}$ is ratio of modules.

- mating condition is

$$\sin\left(\frac{\pi}{k}\right) > \frac{z_2 + 2}{z_1 + z_2}, \qquad \sin\left(\frac{\pi}{k}\right) > \frac{z_2' + 2}{z_3 - z_2'};$$

- coincidence condition is

$$\frac{z_3}{k}\,(1 - u_{31}^{í}) = \gamma,$$

where γ is any integer numeral;

- condition of right engagement (when $\alpha = 20^{o}$ and $h_a^* = 1$) is

$$z_1 \geq 17,\ \ z_2 \geq 17,\ \ z_2' \geq 20,\ z_3 \geq 85,\ \ z_3 - z_2' \geq 8.$$

1.3. Set up equations for determination of the number of teeth of the planetary gearing toothed wheels. For that we make the following designations:

$$\frac{z_2'}{z_2} = x, \qquad \frac{z_2'}{z_3} = y.$$

Parameter y can be also found according to the coaxiality condition:

$$y = \frac{(u_{1H} - 1)x - g}{(u_{1H} - 1)(x + g)}.$$

Equations for determination of the number of teeth of the planetary gearing toothed wheels have the following form

$$z_1 : z_2 : z_2' : z_3 : \gamma = \frac{k}{y(u_{1H} - 1)} : k : kx : \frac{kx}{y} : \frac{(u_{1H} - 2)x}{y(u_{1H} - 1)}.$$

These equations provide carrying out the coaxiality condition and the coincidence condition when the velocity ratio of the planetary gearing is known. That is why after choosing gears number of teeth it is necessary to check for the mating condition and for the condition of right engagement.

1.4. Assume $x = \frac{1}{3}$ and q=1 (the magnitude of x is not recommended to be chosen as 1). Then it is possible to calculate y:

$$y=\frac{\left(u_{1H}-1\right)x-g}{\left(u_{1\text{í}}-1\right)\left(x+g\right)}=\frac{\left(14-1\right)\frac{1}{3}-1}{\left(14-1\right)\left(\frac{1}{3}+1\right)}=\frac{5}{26}.$$

1.5. Number of teeth of the planetary gearing toothed wheels may be calculated in the following way:

$$z_1:z_2:z_2':z_3:\gamma=\frac{k}{y\left(u_{1H}-1\right)}:k:kx:\frac{kx}{y}:\frac{\left(u_{1H}-2\right)x}{y\left(u_{1H}-1\right)},$$

$$z_1:z_2:z_2':z_3:\gamma=\frac{3}{\frac{5}{26}\cdot\left(14-1\right)}:3:3\cdot\frac{1}{3}:\frac{3\cdot\frac{1}{3}}{\frac{5}{26}}:\frac{\frac{1}{3}\cdot\left(14-2\right)}{\frac{5}{26}\cdot\left(14-1\right)},$$

$$z_1:z_2:z_2':z_3:\gamma=\frac{6}{5}:3:1:\frac{26}{5}:\frac{8}{5}.$$

As $\gamma=\frac{8}{5}$ should be an integer numeral we multiply all right hand components of the last equality by 20. Then

$$z_1:z_2:z_2':z_3:\gamma=24:60:20:104:32.$$

Consequently

$$z_1=24,\ z_2=60,\ z_2'=20,\ z_3=104.$$

1.6. Check the mating condition for planet pinions:

$$\sin\left(\frac{\pi}{k}\right)>\frac{z_2+2}{z_1+z_2},\qquad \sin\left(\frac{\pi}{k}\right)>\frac{z_2'+2}{z_3-z_2'},$$

$$\sin\left(\frac{\pi}{3}\right)=\sin\left(\frac{180}{3}\right)=\sin 60^\circ=0.866,$$

$$\frac{z_2+2}{z_1+z_2}=\frac{60+2}{24+60}=0.7381,\qquad \frac{z_2'+2}{z_3-z_2'}=\frac{20+2}{104-20}=0.2619.$$

The mating condition is carried out because $\sin 60^\circ > 0.7381$.

1.7. Check the condition of right engagement:

$$z_1 \geq 17,\ \ z_2 \geq 17,\ z_2' \geq 20,\ z_3 \geq 85,\ \ z_3 - z_2' \geq 8.$$

In our case

$$z_1 = 24 > 17,\ \ z_2 = 60 > 17,\ \ z_2' = 20 \geq 20,\ \ z_3 = 104 > 85,$$

$$z_3 - z_2' = 104 – 20 = 84 > 8.$$

Thus, the condition of right engagement is fulfilled.

If the mating condition and the condition of right engagement are not right it is necessary to recalculate the number of teeth with other magnitude of *x*.

1.8. Check the velocity ratio

$$u_{1H} = 1 - u_{13}^{f} = 1 + \frac{z_2 z_3}{z_1 z_2'} = 1 + \frac{60 \cdot 104}{24 \cdot 20} = 14.$$

The number of teeth of the planetary gearing toothed wheels $z_1 = 24$, $z_2 = 60$, $z_2' = 20$, $z_3 = 104$ was determined correctly. They provide given velocity ratio of the mechanism and fulfilling all necessary conditions.

2. Determination of gears diameters:

$$d_{w1} = d_1 = m \cdot z_1 = 3 \cdot 24 = 72 \text{ mm},$$

$$d_{w2} = d_2 = m \cdot z_2 = 3 \cdot 60 = 180 \text{ mm},$$

$$d_{w2}' = d_2' = m \cdot z_2' = 3 \cdot 20 = 60 \text{ mm},$$

$$d_{w3} = d_3 = m \cdot z_3 = 3 \cdot 104 = 312 \text{ mm}.$$

3. Kinematic analysis of the planetary gearing. It is made in the same way as in example 1.

Example 3

Determine the number of teeth of toothed wheels and carry out kinematic analysis for the planetary gear speed reducer shown in Fig.1.8 if number of planet pinions k = 4; rotational speed of the input shaft n_1 = 1200 rpm; rotational speed of the output shaft n_H = 300 rpm; module of planetary gearing toothed wheels m = 3 mm. Number of teeth of gears must satisfy to the following condition $17 \leq z \leq 180$. Toothed wheels have to be manufactured by standard cutter with $\alpha = 20^{o}$ and $h_a^* = 1$.

Solution

1. Determination of the number of teeth of the planetary gearing toothed wheels.

1.1. Determine the velocity ratio of the planetary gearing u_{14} taking into account the fact that driver H rotates to opposite side with respect to movable sun gear 1. Consequently, this velocity ratio should be negative and it is found as

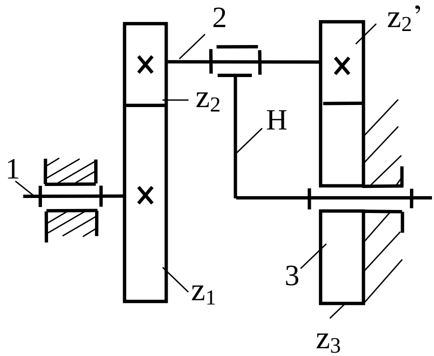


Fig.1.8. Planetary gearing with double planet pinions

$$u_{1H} = 1 - u_{13}^{\acute{I}} = 1 - \frac{z_2 z_3}{z_1 z_2'} =$$

$$= -\frac{n_1}{n_H} = -\frac{1200}{300} = -4,$$

where $u_{13}^{\acute{I}} = \dfrac{z_2 z_3}{z_1 z_2'}$ is the velocity ratio of the reversed mechanism.

1.2. Write the main conditions for the planetary gearing:

- coaxiality condition is

$$m_{12} \cdot (z_1 + z_2) = m_{2'3} \cdot (z_2' + z_3),$$

where m_{12} is a module of gears z_1 and z_2; $m_{2'3}$ is a module of gears z_2' and z_3. This condition may be written in the following way

$$q \cdot (z_1 + z_2) = (z_3 - z_2'),$$

where $q = \dfrac{m_{12}}{m_{2'3}}$ is ratio of modules.

- mating condition is

$$\sin\left(\frac{\pi}{k}\right) > \frac{z_2 + 2}{z_1 + z_2}, \qquad \sin\left(\frac{\pi}{k}\right) > \frac{z_2' + 2}{z_2' + z_3};$$

- coincidence condition is

$$\frac{z_3}{k}\cdot\left(1-u_{31}^{H}\right)=\gamma\,,$$

where γ is any integer numeral;

- condition of right engagement (when $\alpha = 20^{o}$ and $h_a^{*} = 1$) is

$$z_1 \geq 17,\ \ z_2 \geq 17,\ \ z_2{'} \geq 17,\ z_3 \geq 17.$$

1.3. Set up equations for determination of the number of teeth of the planetary gearing toothed wheels. For this purpose we introduce the following designations:

$$\frac{z_2^{'}}{z_2}=x\,,\qquad \frac{z_2^{'}}{z_3}=y\,.$$

Parameter y can be also calculated according to the coaxiality condition:

$$y=\frac{gu_{31}^{Í}-x}{x-g}\,.$$

Values of q and x have to satisfy to the following condition

$$g\,u_{31}^{Í}< \ x<g.$$

Equations for determination of the number of teeth of the planetary gearing toothed wheels are

$$z_1 : z_2 : z_2' : z_3 : \gamma=\frac{k(g-x)}{1-u_{1H}}:k\left(x-\frac{g}{1-u_{1H}}\right):kx\left(x-\frac{g}{1-u_{1H}}\right):kx(g-x):$$

$$:x(g-x)\frac{u_{1H}}{u_{1H}-1}$$

These equations provide carrying out the coaxiality condition and the assembling condition when the velocity ratio of the planetary gearing is known. That is why after choosing number of teeth of toothed wheels it is necessary to check for the mating condition and for the condition of right engagement.

1.4. Assume $x=\frac{2}{5}$ and q =1 (the magnitude of x is not recommended to choose as 1) and check the condition

$$g u_{31}^{H} < x < g.$$

For this purpose we will find u_{31}^{H}

$$u_{31}^{Í} = \frac{1}{u_{13}^{Í}} = \frac{1}{1-u_{1H}} = \frac{1}{1-(-4)} = \frac{1}{5}.$$

After substituting q, x and u_{31}^{H} we obtain

$$1 \cdot \frac{1}{5} < \frac{2}{5} < 1.$$

Thus, values of q and x satisfy to required inequality.

1.5. Determine number of teeth of the planetary gearing toothed wheels:

$$z_1 : z_2 : z_2' : z_3 : \gamma = \frac{k(g-x)}{1-u_{1H}} : k\left(x - \frac{g}{1-u_{1H}}\right) : kx\left(x - \frac{g}{1-u_{1H}}\right) : kx(g-x) :$$

$$: x(g-x)\frac{u_{1H}}{u_{1H}-1},$$

$$z_1 : z_2 : z_2' : z_3 : \gamma = \frac{4\cdot\left(1-\frac{2}{5}\right)}{1-(-4)} : 4\cdot\left(\frac{2}{5} - \frac{1}{1-(-4)}\right) : 4\cdot\frac{2}{5}\cdot\left(\frac{2}{5} - \frac{1}{1-(-4)}\right) :$$

$$: 4\cdot\frac{2}{5}\cdot\left(1-\frac{2}{5}\right) : \frac{2}{5}\cdot\left(1-\frac{2}{5}\right)\cdot\frac{-4}{-4-1},$$

$$z_1 : z_2 : z_2' : z_3 : \gamma = \frac{12}{25} : \frac{4}{5} : \frac{8}{25} : \frac{24}{25} : \frac{24}{125}.$$

As $\gamma = \frac{24}{125}$ should be an integer numeral we multiply all right hand components of the last equality by $\frac{125}{2}$. Then

$$z_1 : z_2 : z_2' : z_3 : \gamma = 30 : 50 : 20 : 60 : 12.$$

Consequently

$$z_1 = 30,\ z_2 = 50,\ z_2' = 20,\ z_3 = 60.$$

1.6. Check the mating condition for planet pinions:

$$\sin\left(\frac{\pi}{k}\right) > \frac{z_2+2}{z_1+z_2}, \qquad \sin\left(\frac{\pi}{k}\right) > \frac{z_2'+2}{z_3+z_2'};$$

$$\sin\left(\frac{\pi}{4}\right) = \sin\left(\frac{180}{4}\right) = \sin 45^\circ = 0.7071,$$

$$\frac{z_2+2}{z_1+z_2} = \frac{50+2}{30+50} = 0.65, \qquad \frac{z_2'+2}{z_2'+z_3} = \frac{20+2}{20+60} = 0.275.$$

The mating condition is carried out because $\sin 45^\circ > 0.65$.

1.7. Check the condition of right engagement:

$$z_1 \geq 17,\ \ z_2 \geq 17,\ z_2' \geq 17,\ z_3 \geq 17.$$

In our case

$$z_1 = 30 > 17,\ \ z_2 = 50 > 17,\ \ z_2' = 20 > 17,\ \ z_3 = 60 > 17.$$

Thus, the condition of right engagement is fulfilled.

If the mating condition and the condition of right engagement are not right it is necessary to recalculate the number of teeth with other magnitude of *x*.

1.8. Check the velocity ratio

$$u_{1H} = 1 - u_{13}^{H} = 1 - \frac{z_2 z_3}{z_1 z_2'} = 1 - \frac{50 \cdot 60}{30 \cdot 20} = -4.$$

The number of teeth of the planetary gearing toothed wheels $z_1 = 30$, $z_2 = 50$, $z_2' = 20$, $z_3 = 60$ was determined correctly. They provide given velocity ratio of the mechanism and fulfilling all necessary conditions.

2. Determination of gears diameters:

$$d_{w1} = d_1 = m \cdot z_1 = 3 \cdot 30 = 90 \text{ mm},$$

$$d_{w2} = d_2 = m \cdot z_2 = 3 \cdot 50 = 150 \text{ mm},$$

$$d_{w2}' = d_2' = m \cdot z_2' = 3 \cdot 20 = 60 \text{ mm},$$

$$d_{w3} = d_3 = m \cdot z_3 = 3 \cdot 60 = 180 \text{ mm}.$$

3. Kinematic analysis of the planetary gearing. It is made in the same way as in example 1.

2. DESIGNING INVOLUTE GEARING

2.1 Theoretical information

Lateral profile of gear teeth can be outlined by different curves. Nowadays gears with involute profile of teeth are mainly used in the mechanical engineering.

There are two methods by means of which gears with involute teeth profile (involute gears): *formed cutter or copying method* and *generating method.* In the first case end and side mills are used as a tool. This method allows to cut only gears in which tooth thickness measured along the nominal pitch circle is equal to the space width along the same circle. Besides, accuracy of gears manufacturing is not high by means of this method.

Generating method is more universal method because it allows to cut gears with different number of teeth by the same cutting tool. Besides, we have possibility to produce both standard and non-standard (modified) gears depending upon disposition of a cutting tool (gear cutter, rack cutter or hob cutter) with respect to a blank (Fig.2.1).

If the module line of the rack cutter is a tangent to the nominal pitch circle of the gear being cut we obtain a *standard gear*.

If the rack module line is removed from the nominal pitch circle of the gear being cut or intersects the latter *nonstandard* or *modified gears* are obtained.

Distance **b** between the module line of the rack cutter and the nominal pitch circle of the gear being cut is called *rack offset*. The ratio of the rack offset to the module of the gear is called *offset factor* **x**

$$\mathrm{x} = \frac{\mathrm{b}}{m}.$$

Offset factor can be positive, negative or equals to zero.

When the rack offset b = 0, the offset factor x = 0 and we deal with standard also called zero-offset gears. In this case tooth thickness **s** of the gear measured along the nominal pitch circle is equal to space width **e** along the same circle.

$$\mathrm{s} = \mathrm{e} = \frac{\pi \cdot m}{2}.$$

When the rack cutter is removed from the blank, b>0, x is positive and gear being cut is called positive-offset gear. For positive-offset gears

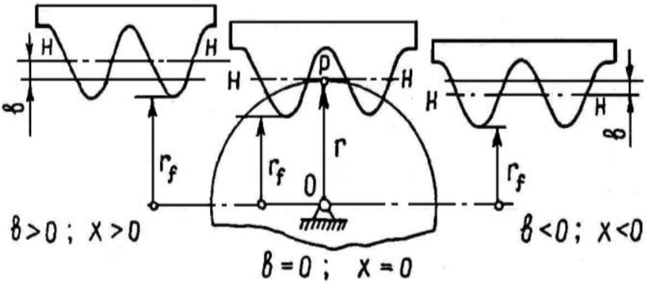


Fig 2.1. Disposition of the rack-cutter during production of a gear

tooth thickness measured along the nominal pitch circle is greater than space width along the same circle (s>e; $s > \frac{\pi \cdot m}{2}$).

The radius of the positive-offset gear root circle $\mathbf{r_f}$ is increased by the magnitude of the rack offset $b = x \cdot m$ in comparison with zero-offset gears. That is why

$$d_f = m \cdot (z - 2{,}5 + 2 \cdot x). \qquad (2.1)$$

When the rack cutter module line intersects the nominal pitch circle of a gear being cut, b<0, offset factor **x** is negative and obtained gears are called as negative-offset gears. For negative-offset gears

$$s<e;\ s < \frac{\pi \cdot m}{2}.$$

Diameter of the root circle of a negative-offset gear is calculated by formula (2.1) but offset factor **x** has to be substituted with sign “minus”.

Let us determine a tooth thickness along the nominal pitch circle of nonstandard (modified) gears. For this purpose we remove a rack cutter from the centre of a blank by the distance $b = x \cdot m$ (Fig.2.2). In this case the space width of the rack cutter measured along straight line that is a tangent to the nominal pitch circle of a gear being cut is increased by the magnitude $2\Delta = 2 \cdot x \cdot m \cdot \mathrm{tg}\alpha$. Tooth thickness of a gear measured along the nominal pitch circle is increased by the same distance. That is why

$$s = \frac{\pi \cdot m}{2} + 2 \cdot x \cdot m \cdot \mathrm{tg}\,\alpha = m \cdot \left(\frac{\pi}{2} + 2 \cdot x \cdot \mathrm{tg}\alpha \right). \qquad (2.2)$$

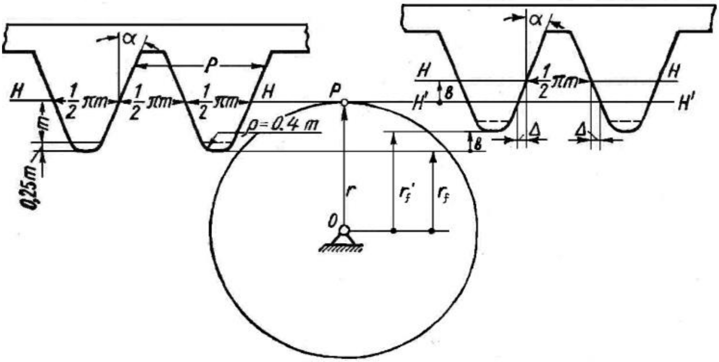


Fig 2.2. Determination of a tooth thickness measured along the nominal pitch circle of a non-standard gear

Let us determine a tooth thickness $\mathbf{s_y}$ measured along a circle of arbitrary radius $\mathbf{r_y}$.

From Fig. 2.3 we can see that

$$\gamma_y + \theta_y = \gamma + \theta \,. \tag{2.3}$$

Angles $\boldsymbol{\gamma_y}$ and $\boldsymbol{\gamma}$ are determined from the following formulas

$$\gamma_y = \frac{s_y}{2 \cdot r_y} = \frac{s_y}{m_y \cdot z}; \qquad \gamma = \frac{s}{m \cdot z},$$

where $\boldsymbol{m_y}$ and $\boldsymbol{m}$ are modules along circles of radii $\mathbf{r_y}$ and $\mathbf{r}$.

After substituting $\boldsymbol{\gamma_y}$ and $\boldsymbol{\gamma}$ to (2.3) and taking into account that $\theta_y = inv\alpha_y$ we obtain

$$\frac{s_y}{m_y \cdot z} + inv\alpha_y = \frac{s}{m \cdot z} + inv\alpha,$$

whence

$$\frac{s_y}{m_y} = \frac{s}{m} + z \cdot (inv\alpha - inv\alpha_y).$$

After substituting value of s from (2.2) to the last formula we obtain

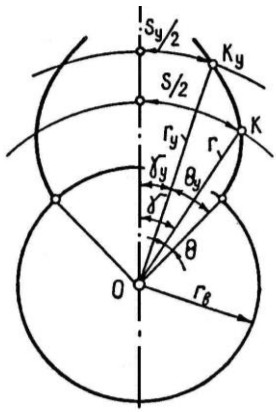


Fig 2.3. Determination of a tooth thickness along any circle

$$s_y = m_y \cdot \left[\frac{\pi}{2} + 2 \cdot x \cdot tg\,\alpha + z \cdot \left(inv\alpha - inv\alpha_y\right)\right], \qquad (2.4)$$

where

$$\cos\alpha_y = \frac{r_b}{r_y}.$$

Formula (2.4) allows to determine a tooth thickness measured along a circle of arbitrary radius r_y. For example, tooth thickness measured along the addendum circle is determined as

$$s_a = m_a \cdot \left[\frac{\pi}{2} + 2 \cdot x \cdot tg\,\alpha + z \cdot \left(inv\alpha - inv\alpha_a\right)\right],$$

where $m_a = \dfrac{d_a}{z}$, $\cos\alpha_a = \dfrac{d_b}{d_a}$.

Nonstandard gearings

The nonstandard gearing may be made up of two nonstandard gears or nonstandard and standard gears.

Depending upon the total offset factor $x_\Sigma = x_1 + x_2$ we will distinguish between gearings with $x_\Sigma = 0$ and gearings with $x_\Sigma \neq 0$.

Let us consider a gearing with $x_\Sigma \neq 0$ which consists of gears 1 and 2 of numbers of teeth $\mathbf{z_1}$ and $\mathbf{z_2}$ and offset factors $\mathbf{x_1}$ and $\mathbf{x_2}$ correspondingly (Fig.2.4).

In order to determine the pressure angle $\boldsymbol{\alpha_w}$ of the gearing it is necessary to find tooth thicknesses of gears 1 and 2 measured along pitch circles. For this purpose we use formula (2.4)

$$s_{w1} = m_w \cdot \left[\frac{\pi}{2} + 2 \cdot x_1 \cdot tg\alpha + z_1 \cdot (inv\alpha - inv\alpha_w)\right], \qquad (2.5)$$

$$s_{w2} = m_w \cdot \left[\frac{\pi}{2} + 2 \cdot x_2 \cdot tg\alpha + z_2 \cdot (inv\alpha - inv\alpha_w)\right], \qquad (2.6)$$

where $\boldsymbol{m_w}$ is module measured along the pith circle.

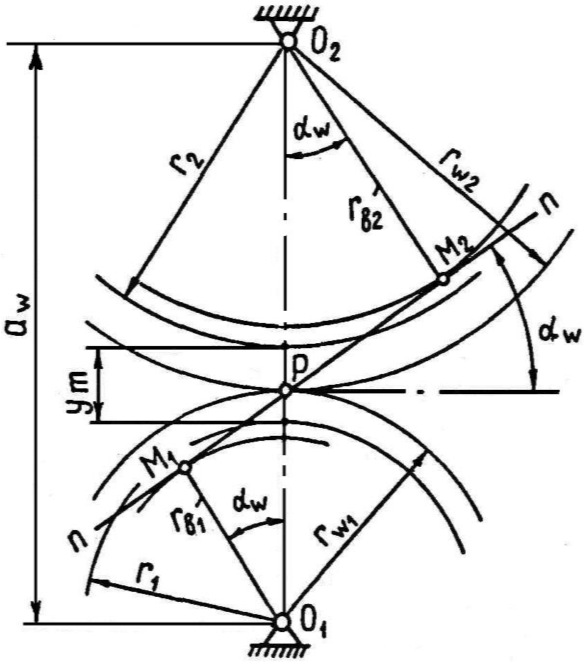


Fig. 2.4. Parameters of a non-standard involute gearing

To provide engagement of gears without backlash a tooth thickness along the pitch circle of one gear has to be equal to a space width of the other gear. That is why

$$s_{w1} + s_{w2} = p_w = \pi \cdot m_w \tag{2.7}$$

After adding left and right parts of (2.5) and (2.6) and taking into account (2.7) we obtain

$$inv\alpha_w = \frac{2 \cdot x_\Sigma}{z_\Sigma} \cdot tg\alpha + inv\alpha,$$

where $x_\Sigma = x_1 + x_2$ and $z_\Sigma = z_1 + z_2$.

If $x_\Sigma = 0$, we have $\alpha_w = \alpha = 20°$.

Let us determine centre distance $\boldsymbol{a}_w$ of a nonstandard gearing. From Fig 2.4 we have

$$a_w = r_{w1} + r_{w2} = \frac{r_{b1} + r_{b2}}{\cos\alpha_w}. \tag{2.8}$$

For a standard (normal) gearing in which radii of pitch circles are equal to corresponding radii of nominal pitch circles ($r_{w1} = r_1$, $r_{w2} = r_2$) and $\alpha_w = \alpha$, a centre distance is determined as

$$a_0 = r_1 + r_2 = \frac{r_{b1} + r_{b2}}{\cos\alpha} = \frac{m \cdot z_\Sigma}{2}. \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$a_w = a_0 \cdot \frac{\cos\alpha}{\cos\alpha_w} = \frac{m \cdot z_\Sigma \cdot \cos\alpha}{2 \cdot \cos\alpha_w}.$$

If $x_\Sigma = 0$, we have

$$a_w = a_0 = \frac{m \cdot z_\Sigma}{2}$$

and pitch circles coincide with nominal pitch circles.

If $x_\Sigma \neq 0$, nominal pitch circles of gears are removed from each other by distance ***ym*** (Fig.2.4). This distance is called perceived offset and **y** is perceived offset factor which is calculated as

$$y = \frac{a_w - a_0}{m} = \frac{z_\Sigma}{2} \cdot \left(\frac{\cos\alpha}{\cos\alpha_w} - 1 \right).$$

Let us deduce a formula for determination of addendum circle diameter of a nonstandard gear.

While designing an involute gearing two requirements have to be carried out:

- teeth of gears have to be engaged with each other without a backlash;
- standard radial clearance $c = 0.25 \cdot m$ has to be made between addendum circle of one gear and root circle of the other one.

The first requirement is carried out if

$$a_w = r_1 + r_2 + y \cdot m. \tag{2.10}$$

According to the second requirement

$$a_w = r_{a2} + c + r_{f2}. \qquad (2.11)$$

After solving (2.10) and (2.11) jointly and taking into account that $r_1 = 0.5\cdot m\cdot z_1$, $r_2 = 0.5\cdot m\cdot z_2$, $r_{f2} = 0.5\cdot m\cdot\left(z_2 - 2.5 + 2\cdot x_2\right)$, $c=0.25\cdot m$ we obtain

$$d_{a2} = m\cdot\left(z_1 + 2 + 2\cdot x_1 - 2\cdot\Delta y\right),$$

where $\Delta y = x_\Sigma - y$ is called equalized offset factor.

Equalized offset $\boldsymbol{\Delta y\cdot m}$ is introduced to obtain an involute gearing without backlash and with standard value of the radial clearance.

2.2.Geometrical calculation of involute gearing of external contact

Initial data

Make geometrical calculation of an involute gearing of external contact if centre distance $a_w = 70$ mm, number of teeth of the pinion $z_1 = 15$, number of teeth of the gear $z_2 = 30$, module $m = 3$ mm, offset factors ratio $\frac{x_1}{x_2} = 1.55$ ($\alpha = 20^\circ$; $c^* = 0.25$; $h_a^* = 1$).

Solution

1. Determine the total number of teeth

$$z_\Sigma = z_1 + z_2 = 15 + 30 = 45.$$

2. Calculate the pressure angle

$$\cos\alpha_w = \frac{m\cdot z_\Sigma\cdot\cos\alpha}{2\cdot a_w} = \frac{3\cdot 45\cdot 0.94}{2\cdot 70} = 0.9064;$$

$$\alpha_w = 0.4361\,\text{rad} = 24^\circ 59'; \ \operatorname{tg}\alpha_w = 0.466.$$

$$inv\alpha_w = \operatorname{tg}\alpha_w - \alpha_w = 0.466 - 0.4361 = 0.0299.$$

3. Find the total offset factor

$$x_\Sigma = \frac{z_\Sigma\cdot\left(inv\alpha_w - inv\alpha\right)}{2\cdot\operatorname{tg}\alpha} = \frac{45\cdot\left(0.0299 - 0.0149\right)}{2\cdot 0.364} = 0.927.$$

4. Determine offset factors for the pinion and the gear taking into account that $\frac{x_1}{x_2} = 1.55$.

The pinion offset factor is

$$x_1 = 1.55 \cdot x_2.$$

On the other hand $x_1 + x_2 = x_\Sigma$. That is why

$$x_2 = \frac{x_\Sigma}{2.55} = \frac{0.927}{2.55} = 0.364,$$

$$x_1 = x_\Sigma - x_2 = 0.927 - 0.364 = 0.563.$$

Taking into account that $z_1 < 17$ the following condition should be carried out to eliminate a tooth undercutting

$$x_1 \geq x_{1\min},$$

where

$$x_{1\min} = \frac{17 - z_1}{17} = \frac{17 - 15}{17} = 0.1176.$$

The condition is carried out.

5. Calculate the perceived offset factor

$$y = \frac{a_w - 0.5 \cdot m \cdot z_\Sigma}{m} = \frac{70 - 0.5 \cdot 3 \cdot 45}{3} = 0.83.$$

6. Determine the equalized offset factor

$$\Delta y = x_\Sigma - y = 0.927 - 0.83 = 0.097.$$

7. Determine diameters of nominal pitch circles

$$d_1 = m \cdot z_1 = 3 \cdot 15 = 45 \text{ mm},$$

$$d_2 = m \cdot z_2 = 3 \cdot 30 = 90 \text{ mm}.$$

8. Find diameters of base circles

$$d_{b1} = m \cdot z_1 \cdot \cos\alpha = 3 \cdot 15 \cdot \cos 20 = 42.3 \text{ mm},$$

$$d_{b2} = m \cdot z_2 \cdot \cos\alpha = 3 \cdot 30 \cdot \cos 20 = 84.6 \text{ mm}.$$

9. Determine diameters of pitch circles

$$d_{w1} = m \cdot z_1 \cdot \frac{\cos\alpha}{\cos\alpha_w} = 3 \cdot 15 \cdot \frac{0.94}{0.9064} = 46.668 \text{ mm},$$

$$d_{w2} = m \cdot z_2 \cdot \frac{\cos\alpha}{\cos\alpha_w} = 3 \cdot 30 \cdot \frac{0.94}{0.9064} = 93.336 \text{ mm}.$$

10. Check the center distance

$$a_w = \frac{d_{w1} + d_{w2}}{2} = \frac{46.668 + 93.336}{2} = 70.002 \text{ mm}.$$

11. Calculate diameters of addendum circles

$$d_{a1} = m \cdot (z_1 + 2 + 2 \cdot x_1 - 2 \cdot \Delta y) = 3 \cdot \left(15 + 2 + 2 \cdot 0.563 - 2 \cdot 0.097\right) = 53.796 \text{ mm},$$

$$d_{a2} = m \cdot (z_2 + 2 + 2 \cdot x_2 - 2 \cdot \Delta y) = 3 \cdot \left(30 + 2 + 2 \cdot 0.364 - 2 \cdot 0.097\right) = 97.602 \text{ mm}.$$

12. Determine diameters of root circles

$$d_{f1} = m \cdot \left(z_1 - 2.5 + 2 \cdot x_1\right) = 3 \cdot \left(15 - 2.5 + 2 \cdot 0.563\right) = 40.878 \text{ mm},$$

$$d_{f2} = m \cdot \left(z_2 - 2.5 + 2 \cdot x_2\right) = 3 \cdot \left(30 - 2.5 + 2 \cdot 0.364\right) = 84.684 \text{ mm}.$$

13. Check the correctness of previous calculations:

$$a_w = \frac{d_{a1}}{2} + \frac{d_{f2}}{2} + 0.25 \cdot m = \frac{53.796}{2} + \frac{84.684}{2} + 0.25 \cdot 3 = 69.99 \text{ mm},$$

$$a_w = \frac{d_{a2}}{2} + \frac{d_{f1}}{2} + 0.25 \cdot m = \frac{97.602}{2} + \frac{40.878}{2} + 0.25 \cdot 3 = 69.99 \text{ mm}.$$

14. Determine the whole depth of the tooth

$$h = 2.25 \cdot m - \Delta y \cdot m = 2.25 \cdot 3 - 0.097 \cdot 3 = 6.459 \text{ mm},$$

$$h = \frac{d_{a1} - d_{f1}}{2} = \frac{53.796 - 40.878}{2} = 6.459 \text{ mm},$$

$$h=\frac{d_{a2}-d_{f2}}{2}=\frac{97.602-84.684}{2}=6.459\text{ mm}.$$

15. Find the circular pitch

$$p_w=\pi\cdot m\cdot\frac{\cos\alpha}{\cos\alpha_w}=\frac{3.14\cdot 3\cdot 0.94}{0.9064}=9.769\text{ mm}.$$

16. Determine tooth thicknesses measured along nominal pitch circles

$$s_1=m\cdot\left(\frac{\pi}{2}+2\cdot x_1\cdot \mathrm{tg}\,\alpha\right)=3\cdot\left(\frac{3.14}{2}+2\cdot 0.563\cdot 0.364\right)=5.94\text{ mm},$$

$$s_2=m\cdot\left(\frac{\pi}{2}+2\cdot x_2\cdot \mathrm{tg}\,\alpha\right)=3\cdot\left(\frac{3.14}{2}+2\cdot 0.364\cdot 0.364\right)=5.5\text{ mm}.$$

17. Calculate tooth thicknesses measured along pitch circles

$$s_{w1}=d_{w1}\cdot\left(\frac{s_1}{d_1}+inv\,\alpha-inv\,\alpha_w\right)=$$

$$=46.668\cdot\left(\frac{5.94}{45}+0.0149-0.0299\right)=5.46\text{ mm},$$

$$s_{w2}=d_{w2}\cdot\left(\frac{s_2}{d_2}+inv\,\alpha-inv\,\alpha_w\right)=$$

$$=93.336\cdot\left(\frac{5.5}{90}+0.0149-0.0299\right)=4.304\text{ mm}.$$

18. Calculate tooth thicknesses measured along base circles

$$s_{b1}=d_{b1}\cdot\left(\frac{s_1}{d_1}+inv\,\alpha\right)=42.3\cdot\left(\frac{5.94}{45}+0.0149\right)=6.214\text{ mm},$$

$$s_{b2}=d_{b2}\cdot\left(\frac{s_2}{d_2}+inv\,\alpha\right)=84.6\cdot\left(\frac{5.5}{90}+0.0149\right)=6.431\text{ mm}.$$

19. Determine tooth thicknesses measured along addendum circles taking into account that we know pressure angles α_{a1}, α_{a2} for involute points through which addendum circles pass. These angles are calculated in the following way:

$$\cos\alpha_{a1} = \frac{d_{b1}}{d_{a1}} = \frac{42.3}{53.796} = 0.786,$$

$$\alpha_{a1} = 0.667 \text{ rad} = 38^{\circ}\ 09', \ \text{tg}\ \alpha_{a1} = 0.787,$$

$$inv\alpha_{a1} = \text{tg}\alpha_{a1} - \alpha_{a1} = 0.787 - 0.667 = 0.12$$

$$\cos\alpha_{a2} = \frac{d_{b2}}{d_{a2}} = \frac{84.6}{97.602} = 0.867,$$

$$\alpha_{a2} = 0.522 \text{ rad} = 29^{\circ}\ 53', \ \text{tg}\ \alpha_{a1} = 0.575,$$

$$inv\alpha_{a2} = \text{tg}\alpha_{a2} - \alpha_{a2} = 0.575 - 0.522 = 0.053.$$

Consequently,

$$s_{a1} = d_{a1} \cdot \left(\frac{s_1}{d_1} + inv\alpha - inv\alpha_{a1} \right) =$$

$$= 53.796 \cdot \left(\frac{5.94}{45} + 0.0149 - 0.12 \right) = 1.447 \text{ mm},$$

$$s_{a2} = d_{a2} \cdot \left(\frac{s_2}{d_2} + inv\alpha - inv\alpha_{a2} \right) =$$

$$= 97.602 \cdot \left(\frac{5.5}{90} + 0.0149 - 0.053 \right) = 2.246 \text{ mm}.$$

20. Check a tooth for sharpening. According to this condition a tooth thickness measured along an addendum circle should not be less than $0.2 \cdot m$. In our case

$$s_{a1} \geq 0.2 \cdot m = 0.2 \cdot 3 = 0.6 \text{ mm},$$

$$s_{a2} \geq 0.2 \cdot m = 0.2 \cdot 3 = 0.6 \text{ mm}.$$

Consequently, sharpening condition is carried out.

21. Calculate the contact ratio

$$\varepsilon_\alpha = \frac{1}{2\pi} \cdot \left(z_1 \cdot \mathrm{tg}\,\alpha_{a1} + z_2 \cdot \mathrm{tg}\,\alpha_{a2} - z_\Sigma \cdot \mathrm{tg}\,\alpha_w \right) =$$

$$= \frac{1}{2 \cdot 3.14} \cdot (15 \cdot 0.787 + 30 \cdot 0.575 - 45 \cdot 0.466) = 1.287 .$$

The contact ratio must be greater or equal to 1.05 ($\varepsilon_\alpha \geq 1.05$).

2.3. Plotting involute profile of a tooth

1. Choose the length of a segment that represents the centre distance a_w on a drawing. We mark this segment as $\overline{O_1O_2}$. The length of $\overline{O_1O_2}$ has to be greater than 450 mm. The centres of gears O_1 and O_2 may be located outside the drawing. Determine the scale factor μ_l. It should be chosen in such way to obtain a tooth whole depth not less than 40 mm.

Let segment $\overline{O_1O_2}$ be equal to 500 mm. Then the scale factor μ_l is found as

$$\mu_l = \frac{a_w}{\overline{O_1O_2}} = \frac{0.07}{500} = 0.00014 \, \frac{\mathrm{m}}{\mathrm{mm}} .$$

2. Lay off segment $\overline{O_1O_2}$ and through obtained points O_1 and O_2 we draw pitch circles of radii r_{w1} and r_{w2} (Fig.2.5). Point P of their contact is the pitch point. After that we draw base circles of radii r_{b1} and r_{b2} and the general tangent *n-n* to these circles that is the line of action. Points of contact of line *n-n* with the base circles are correspondingly marked as M_1 and M_2. The line of action has to pass through the pitch point P and segments $\overline{O_1M_1}$ and $\overline{O_2M_2}$ must be perpendicular to the first one.

3. Through point P we draw straight line perpendicular to the centre distance $\overline{O_1O_2}$. This line will be the tangent to both pitch circles. The angle between the line of action and the perpendicular to the centre distance is the pressure angle α_w. The magnitude of this angle should be equal to the pressure angle found by the analytical method. In our case $\alpha_w \approx 25^\circ$.

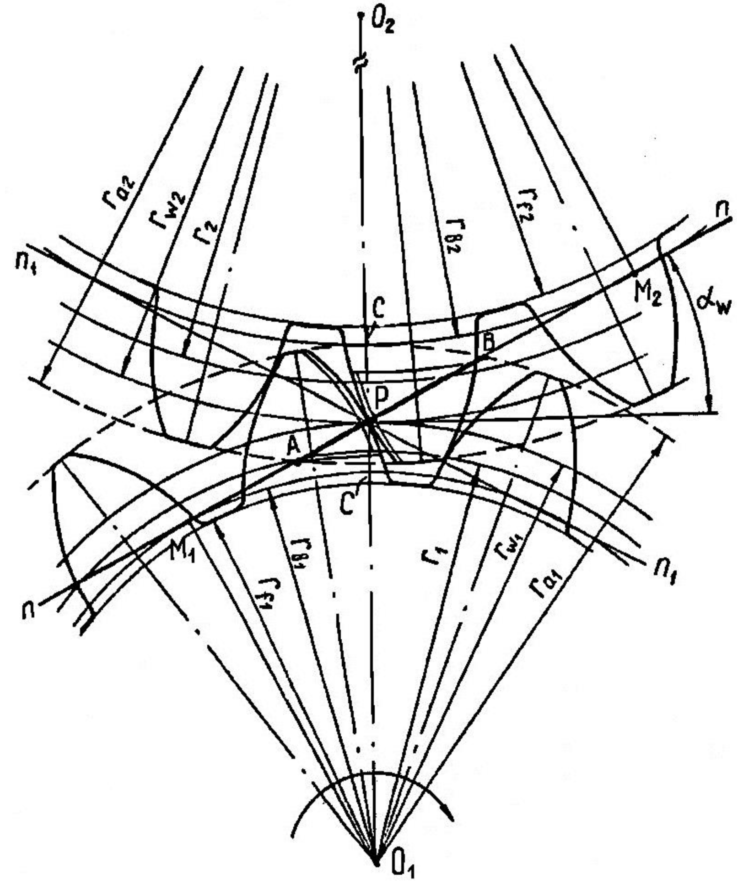


Fig.2.5. Engagement of gears with external contact

4. Draw addendum circles of radii r_{a1} and r_{a2} as well as root circles of radii r_{f1} and r_{f2}. The distance between the addendum circle of one gear and the root circle of the other measured along the centre distance is the radial clearance *c* of a gearing. It is always equal to $0.25 \cdot m$. It is necessary to check if this distance corresponds to $0.25 \cdot m$ on the drawing.

5. Plot a tooth profile of one gear. For this purpose through point P we lay off an arc along the pitch circle whose length is equal to half of the tooth thickness ($s_w/2$). As a result we obtain point 2 that is on the

tooth symmetry axis (Fig 2.6). Through points 2 and O_1 we draw straight line (tooth axis of symmetry) to intersection of this line with the addendum circle, the nominal pitch circle, the base circle at points 1, 3 and 4. After laying off arcs from these points whose lengths are equal to half of tooth thickness measured along corresponding circles we obtain points 1', 3', 4'.If we connect points 1′, P, 3′, 4′ by smooth curve we will obtain right lateral profile of the tooth. In the same way we may find position of points 1′′, 2′′, 3′′, 4′′ that are on the tooth left lateral profile.

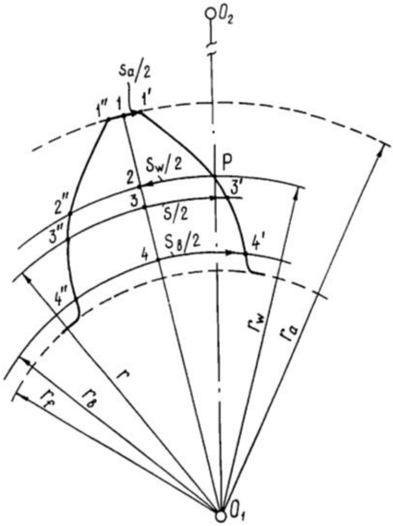


Fig. 2.6. Plotting a tooth involute profile

The tooth lateral surface is described by an involute. The initial point of this curve is on the base circle and inside this circle the involute cannot exist. That is why part of a tooth profile between the base circle and the root circle (when $r_b>r_f$) is described by a straight line parallel to the radius. Obtained linear portion of a tooth profile is joined with the root circle by an arc of radius $\rho = 0.38 \cdot m$. If $r_b<r_f$ the whole lateral profile of a tooth is described by an involute. In this case an involute is joined with the root circle by an arc of the same radius $\rho = 0.38 \cdot m$.

6. Plot a tooth of the second gear with usage of mentioned above method. In this case teeth lateral profiles should contact each other at the pitch point P.

7. Plot two adjacent teeth for gear 1. For that we lay off either arcs of the circular pitch length p_w along the pitch circle or chords of length $\ell_x = d_{w1} \cdot \sin\left(\frac{180°}{z1}\right)$ in both directions relative to the tooth axis of symmetry. As a result we obtain points located on the axes of symmetry of two adjacent teeth. Straight lines connecting these points with the gear centre O_1 represent the axes of symmetry of these teeth. The angle between axes of two adjacent teeth should be equal to $\frac{360°}{z_1}$. Two adjacent teeth are plotted by means of transferring sizes of the already plotted tooth or with usage of the method of templates.

8. Plot two adjacent teeth of gear 2. For this purpose we will use the same method as in the previous point. It is necessary to remember that the angle between axes of two adjacent teeth should be equal to $\frac{360°}{z_2}$.

The drawing of the gearing engagement is shown in Fig. 2.6. Teeth of gear 1 have to contact teeth of gear 2 at three points. These points are located on line *n-n* that is the general tangent to base circles of both gears and passes through the pitch point P. Normal n_1-n_1 to teeth profiles is the line of action for left profiles of teeth of gear 1 and mating profiles of gear 2.

9. Determine working parts of teeth profiles. The most remote points of teeth profiles with respect to centres of gears are points located on the addendum circles. The addendum circle of gear 1 intersects the line of action at point B and the addendum circle of gear 2 intersects line *n-n* at point A. That is why at point A mating profiles are engaged and at point B they are disengaged. Thus, the engagement of gears is carried out within portion AB. This part of the line of action is called a path of contact.

In predetermined direction of rotation only one side of a gear tooth transmits or withstands load. Besides, during operation not the whole tooth profile works and only its certain part. This part is called an active flank. The active flank of gear 1 tooth is limited by a point that is met at point A of the line of action with a point of gear 2 located on the addendum circle. In order to determine position of this point at the tooth profile of gear 1 it is necessary to draw the arc of radius O_1A to intersection of the latter with the tooth profile.

In the same way we can find the limit of the active flank of the tooth for gear 2. On the drawing the active flanks are marked by double lines.

10. Determine the contact ratio by the graphical method:

$$\varepsilon_\alpha = \frac{\overline{AB}}{p_b},$$

where $\overline{AB}$ is the length of the path of contact in mm; p_b is the pitch measured along the base circle in mm.

The error between the contact ratios calculated by analytical method and graphical method should not be greater than 5%.

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