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A LUSTERNIK-SCHNIRELMANN TYPE THEOREM FOR C^1-FRÉCHET MANIFOLDS

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Abstract. We prove a Lusternik-Schnirelmann type theorem for a C^1-function $\varphi : M \to \mathbb{R}$, where M is a connected infinite dimensional Fréchet manifold of class C^1. To this end, in this context we prove the so-called Deformation Lemma and by using it we derive the result generalizing the Minimax Principle.

1. INTRODUCTION

The aim of this paper is to extend the Lusternik-Schnirelmann method to study the minimal numbers of critical points of C^1 real-valued functions on Fréchet manifolds. The Lusternik-Schnirelmann theory is a method to study critical points independent of nondegeneracy considerations which are present in Morse theory. The Morse critical points theory was extended to Hilbert manifolds by Palais [7] and Smale [12]. Also, it was studied in the case of Banach manifolds; cf., Palais [8], Uhlenbeck [15], Tromba [14]. These critical point theories in the context of Hilbert and Banach manifolds provide tools to prove existence theorems in the calculus of variations. For example, they have been used to geodesic problems (cf. the books of Milnor [3] and Palais [9]) and to eigenvalue problems (cf. Browder [1]).

In the case of Fréchet manifolds, a proper intrinsic notion of nondegeneracy that would recover Morse theory has not been defined yet due to natural restrictions. Therefore, it would be desirable to have a Lusternik-Schnirelmann theory in order to apply to problems in the calculus of variation that involve these manifolds.

The classical Lusternik-Schnirelmann theory was extended to Banach manifolds by Palais [6]. The construction of suitable deformations that characterize critical values of functions and satisfy certain conditions is the most technical

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part of his method. The deformation approach is implemented by considering the negative pseudo-gradient flows, so, the existence of pseudo-gradient vector fields is inevitable. Another concern is that the flow generated by a vector field is no longer a global flow since manifolds are infinite dimensional. It was introduced a compactness condition on functions by Palais [7] and Smale [12], known as the Palais-Smale condition, to show that flow generated by the pseudo-gradient field 'descends to the critial point sets' in a strong enough sense. This condition resolves the mentioned issue.

The Palais's approach does not work in full extent for Fréchet manifolds. Because of the deficient topological structures of dual spaces, cotangent bundles do not admit smooth manifold structures. Consequently, the notion of pseudo-gradient vector fields makes no sense. Moreover, the common Palais-Smale condition requires a Finsler structure on a cotangent bundle.

In this paper, to define the Palais-Smale condition on a Fréchet Finsler manifold M we use an auxiliary function Φ. The idea behind considering this function is that on sets where a real-valued function φ on M has no critical points and satisfies the Palais-Smale condition the auxiliary function is negative. This produces an appropriate deformation, Lemma 3.1, which in addition needs to satisfy the following conditions:

A: If $[c-\epsilon, c+\epsilon]$ does not contain critical values, then the topology of a manifold will not change between level sets $c-\epsilon$ and $c+\epsilon$.

B: Starting a little above a critical level c, then we will either bypass an open neighborhood U of the level set c and reach a level $c-\epsilon$ or we will end up in U.

To guarantee the existence of a deformation $\mathcal{H}$ (Corollary 3.5) that satisfies condition **A**, we require that the function φ to be non-constant and closed. The reason for this assumption is that continuous non-constant closed real-valued functions on infinite dimensional Fréchet manifolds are proper ([11, Theorem 1.1]), so strips $\{c-\epsilon \leq \varphi(x) \leq c+\epsilon\}$ are compact. This is a key point in the proof of the existence of such deformations.

In order to the deformation $\mathcal{H}$ satisfies also condition **B** we assume that, in addition, the function φ has finitely many critical critical points. This does not restrict us to prove the main theorem in full generality.

Using the deformation $\mathcal{H}$, we shall derive the standard Minimax Principle (Theorem 3.6) that determines critical values of φ by consideration of minimax expressions of the form $c = \inf \sup \varphi$. Finally, we prove the main theorem (Theorem 3.10) asserting that a non-constant closed C^1- function $\varphi : M \to \mathbb{R}$

which is bounded below and satisfies the Palais-Smale condition on an infinite dimensional connected Fréchet manifold M has at least $\mathrm{Cat}_M M$ critical points.

2. Preliminaries

In this section, we briefly recall the basic concepts of the theory of Fréchet manifolds and establish our notations. We denote by F a Fréchet space whose topology is defined by a sequence $(\| \cdot \|_F^n)_{n\in\mathbb{N}}$ of seminorms which we can always assume to be increasing (by considering $\max_{k\leq n} \| \cdot \|_F^k$, if necessary). Moreover, the complete translation-invariant metric

$$d_F(x,y) := \sum_{n\in\mathbb{N}} \frac{1}{2^n} \cdot \frac{\| x-y \|_F^n}{1+ \| x-y \|_F^n} \tag{2.1}$$

induces the same topology on F. Define a closed unit semi-ball centered at the zero vector 0_F of F by

$$B^n(0_F,1) = \{f \in F : \| f \|_F^n \leq 1\}$$

for each seminorm $\| \cdot \|_F^n$. Let

$$B_\infty(0_F) = \bigcap_{i=1}^{\infty} B^i(0_F,1).$$

The set $B_\infty(0_F)$ is not empty ($0 \in B_\infty(0_F)$) and is infinite (because it is convex so by the Kolmogorov theorem it is bounded only in Banach spaces).

Let E, F be Fréchet spaces, U an open subset of E and $\varphi : U \to F$ a continuous map. Let $CL(E,F)$ be the space of all continuous linear maps from E to F topologized by the compact-open topology. If the directional (Gâteaux) derivatives

$$\mathrm{d}\,\varphi(x)h = \lim_{t\to 0} \frac{\phi(x+th)-\phi(x)}{t}$$

exist for all $x \in U$ and all $h \in E$, and the induced map $\mathrm{d}\,\varphi(x) : U \to CL(E,F)$ is continuous for all $x \in U$, then we say that φ is a Keller's differentiable map of class C^1. The higher directional derivatives and C^k-maps, $k \geq 2$, are defined in the obvious inductive fashion.

Let $k \geq 1$, a C^k-Fréchet manifold is a Hausdorff second countable manifold modeled on a Fréchet space with an atlas of coordinate charts such that the coordinate transition functions are all C^k-maps.

If $\varphi : F \to \mathbb{R}$ at x is C^1, the derivative of φ at x, $\varphi'(x)$, is an element of the dual space F'. The directional derivative of φ at x toward $h \in E$ is given by

$$\mathrm{d}\,\varphi(x)h = \langle \varphi'(x), h \rangle,$$

where $\langle \cdot, \cdot \rangle$ is duality pairing. Let $x \in M$ and $h \in T_x M$. A chart $(x \in U, \psi)$ induces a canonical map ψ_* from $T_x M$ onto F. Let $\varphi : M \to \mathbb{R}$ be a C^1-function, then

$$\varphi'(x,h) = \lim_{t \to 0} \frac{\varphi\big(\psi^{-1}(\varphi(x) + t\psi_*(x)(h))\big) - \varphi(x)}{t}.$$

Definition 2.1. [13] Let F be a Fréchet space T a topological space, and $V = T \times F$ the trivial bundle with fiber F over T. A Finsler structure for V is a collection of continuous functions $\| \cdot \|^n \colon V \to \mathbb{R}^+$, $n \in \mathbb{N}$, such that

(1) For $b \in T$ fixed, $\| (b, f) \|^n = \| f \|_b^n$ is a collection of seminorms on F which gives the topology of F.
(2) Given $k > 1$ and $t_0 \in T$, there exists a neighborhood $\mathcal{U}$ of t_0 such that
$$\frac{1}{k} \| f \|_{t_0}^n \leq \| f \|_u^n \leq k \| f \|_{t_0}^n \tag{2.2}$$
for all $u \in \mathcal{U}$, $n \in \mathbb{N}$, $f \in F$.

Suppose M is a Fréchet manifold modeled on F. Let $\pi_M : TM \to M$ be the tangent bundle and let $\| \cdot \|^n \colon TM \to \mathbb{R}^+$ be a collection of continuous functions, $n \in \mathbb{N}$. We say $\{\| \cdot \|^n\}_{n \in \mathbb{N}}$ is a Finsler structure for TM if for a given $x \in M$ there exists a bundle chart $\psi : U \times F \simeq TM \mid_U$ with $x \in U$ such that

$$\{\| \cdot \|^n \circ \psi^{-1}\}_{n \in \mathbb{N}}$$

is a Finsler structure for $U \times F$.

A Fréchet Finsler manifold is a Fréchet manifold together with a Finsler structure on its tangent bundle. Regular (in particular paracompact) manifolds admit Finsler structures.

If $\{\| \cdot \|^n\}_{n \in \mathbb{N}}$ is a Finsler structure for M then we can obtain a graded Finsler structure, denoted by $(\| \cdot \|^n)_{n \in \mathbb{N}}$, that is $\| \cdot \|^i \leq \| \cdot \|^{i+1}$ for all i.

We define the length of a C^1-curve $\gamma : [a,b] \to M$ with respect to the n-th component by

$$L_n(\gamma) = \int_a^b \| \gamma'(t) \|_{\gamma(t)}^n \, \mathrm{d}\, t.$$

The length of a piecewise path with respect to the n-th component is the sum over the curves constituting to the path. On each connected component of M, the distance is defined by

$$\rho_n(x,y) = \inf_{\gamma} L_n(\gamma),$$

where infimum is taken over all continuous piecewise C^1-curve connecting x to y. Thus, we obtain an increasing sequence of metrics $\rho_n(x,y)$ and define the distance ρ by

$$\rho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\rho_n(x,y)}{1+\rho_n(x,y)}. \tag{2.3}$$

Theorem 2.1. [13] Suppose M is connected and endowed with a Finsler structure $(\|\cdot\|^n)_{n\in\mathbb{N}}$. Then the distance ρ defined by (2.3) is a metric for M. Furthermore, the topology induced by this metric coincides with the original topology of M.

Remark 2.2. The tangent bundle $\pi : TM \to M$ is a C^1-Fréchet manifold modeled on $F \times F$. Thus, we can also define a Finsler structure on the double tangent bundle $\pi_{TM} : T(TM) \to TM$, and it will induce a metric ρ_{TM} on TM compatible with the topology of TM.

3. A Lusternik-Schnirelmann theorem

As mentioned in the introduction, we can define for a Fréchet manifold (non-Banachable) the cotangent bundle as a set to be the dual bundle to the tangent bundle, but it does not admit a smooth manifold structure (see [4, Remark I.3.9]). Thus, we can not define Finsler structures on cotangent bundles and consequently we can not define the usual Palais-Smale compactness condition for differentiable maps by using cotangent bundles. We will define the Palais-Smale condition by using an appropriate auxiliary function.

Let M be a connected C^1-Fréchet manifold, and $\varphi : M \to \mathbb{R}$ a C^1-function. Let $x \in M$, we shall say that x is a critical point of φ if $(\varphi\psi^{-1})'(\psi(x)) = 0$ for a chart $(x \in U, \psi)$ and hence for every chart whose domain contains x.

Let $\{\|\cdot\|_M^n\}_{n\in\mathbb{N}}$ be a Finsler structure on TM. Define a closed unit semi-ball centered at the zero vector 0_x of T_xM by

$$\mathbb{B}^n(0_x, 1) = \{h \in T_xM : \| h \|_x^n \leq 1\}$$

for each $x \in M$ and each seminorm $\|\cdot\|_x^n$. Let

$$\mathbb{B}_\infty(0_x) = \bigcap_{n=1}^{\infty} \mathbb{B}^n(0_x, 1).$$

The set $\mathbb{B}_\infty(0_x)$ is not empty and is infinite because it can be identified with a convex neighborhood of zero of the Fréchet space $U \times F$, where U is an open neighborhood of x.

Let $\varphi : M \to \mathbb{R}$ be a C^1-function and $x \in M$, define

$$\Phi_\varphi(x) = \inf \big\{\varphi'(x,h) : h \in \mathbb{B}_\infty(0_x)\big\}. \tag{3.1}$$

Condition (Palais-Smale). We say that a C^1- function $\varphi : M \to \mathbb{R}$ satisfies the Palais-Smale condition at a level $c \in \mathbb{R}$, $(PS)_c$ in short, in a set $A \subset M$ if each sequence $(m_i)_{i\in\mathbb{N}} \subset A$ such that

$$\varphi(m_i) \to c \quad \text{and} \quad \Phi_\varphi(m_i) \to 0$$

has a convergent subsequence (by the continuity of φ' converges to a critical point).

Let $\varphi : M \to \mathbb{R}$ be a C^1-function. We denote by $\mathrm{Cr}(\varphi)$ the set of critical points of φ, and for $c \in \mathbb{R}$

$$\mathrm{Cr}(\varphi, c) = \{x \in \mathrm{Cr}(\varphi), \varphi(x) = c\},$$

$$\varphi^c = \{x \in M : \varphi(x) \leq c\}.$$

A mapping $\mathcal{H} \in C([0,1] \times M \to M)$ is called a deformation if $\mathcal{H}(0, x) = x$ for all $x \in M$. Let C be a subset of M, we say that $\mathcal{H}$ is a C-invariant for an interval $I \subset [0,1]$ if $\mathcal{H}(t, x) = x$ for all $x \in C$ and all $t \in I$.

A family $\mathcal{F}$ of subset of M is said to be deformation invariant if for each $A \in \mathcal{F}$ and each deformation $\mathcal{H}$ for M, $\mathcal{H}_1(x) :- \mathcal{H}(1, x)$, it follows that

$$\mathcal{H}_1(A) \in \mathcal{F}.$$

The idea of the proof of the following lemma is inspired by Ghoussoub [2, Lemma 1].

Lemma 3.1. Let M be a connected C^1-Fréchet manifold endowed with a complete Finsler metric ρ. Assume $\varphi : M \to \mathbb{R}$ is a C^1-function. Let B and A be closed disjoint subsets of M and let A be compact. Suppose $k > 1$ and $\epsilon > 0$ are such that $\Phi_\varphi(x) < -2\epsilon(1 + k^2)$ for all $x \in A$. Then there exist $t_0 > 0$ and B-invariant deformation $\mathcal{H}$ for $[0, t_0)$ such that

(1) $\rho(\mathcal{H}(t, x), x) \leq kt, \quad \forall x \in M,$
(2) $\varphi(\mathcal{H}(t, x)) - \varphi(x) \leq -2\epsilon(1 + k^2)t, \quad \forall x \in M.$

Proof. Since φ is C^1 and so Gâteaux differentiable, in virtue of the assumption that for $x_i \in A$ we have $\Phi(x_i) < -2\epsilon(1 + k^2)$, we can find a chart $\psi_i : \mathbb{V}_i \to T_{x_i}M$ such that for all $v \in \psi_i(\mathbb{V}_i)$ and a tangent vector $h_i \in \mathbb{B}_\infty(0_{x_i})$ we obtain

$$\langle (\varphi \circ \psi_i^{-1})'(v), h_i \rangle < -2\epsilon(1 + k^2). \tag{3.2}$$

Also, from the Finsler structure on TM we get

$$\frac{1}{k} \parallel \cdot \parallel_u^n \leq \parallel \cdot \parallel_{x_i}^n \leq k \parallel \cdot \parallel_u^n \tag{3.3}$$

for all $n \in \mathbb{N}$ and $u \in \mathbb{V}_i$.

Let $k>1$ be fixed. Let $\mathcal{U}_i \subset \mathbb{V}_i$ be an open neighborhood of x_i such that for some $r_i > 0$

$$B_\rho(\mathcal{U}_i, r_i) \subseteq \mathbb{V}_i, \quad B_{\rho_{TM}}(\psi_i(\mathcal{U}_i), r_i) \subseteq \psi_i(\mathbb{V}_i). \tag{3.4}$$

It follows from the compactness of A that there exists a finite sub-covering $\mathcal{U}_{i_1}, \cdots, \mathcal{U}_{i_p}$. Let $P_{i_1}, \cdots, P_{i_p}$ be a continuous partition of unity subordinated to this sub-covering. For the sake of brevity we write $j \coloneqq i_j$.

Let $\mathrm{b} : M \to [0,1]$ be a continuous function such that $\mathrm{b} \equiv 1$ on A and $\mathrm{b} \equiv 0$ on

$$(M \setminus \bigcup_{j=1}^{p} \mathcal{U}_j) \bigcup B.$$

Let $r_{\min}$ be the minimum of such r_i's and $t_0 = \dfrac{r_{\min}}{1+k^2}$. For $t \in (0, t_0)$ define the function $\sigma_0(t,x) = x$ and the functions $\sigma_j (1 \leq j \leq p)$ inductively by

$$\sigma_j(t,x) = \begin{cases} \psi_j^{-1}\big(\psi_j(\sigma_{j-1}(t,x)) - t\mathrm{b}(x)P_j(x)h_j\big) & \sigma_{j-1}(t,x) \in \mathbb{V}_j \\ \sigma_{j-1}(t,x) & \text{otherwise} \end{cases}$$

where $h_j \in \mathbb{V}_j \cap \mathbb{B}_\infty(0_{x_i})$. For $j=1$, from(3.4) it follows that σ_1 is well defined and continuous. Let x be an arbitrary point in $\mathbb{V}_1$, and for $s \in [0,t]$ let $\jmath_1(s) = \sigma_1(s,x)$ be the curve that joins x to $\sigma_1(t,x)$. Then, for all $n \in \mathbb{N}$

$$\begin{aligned} \rho_n(x, \sigma_1(t,x)) &\leq \int_0^t \parallel \jmath_1'(s) \parallel^n_{\jmath(t)} \mathrm{d}\, s \\ &\leq \int_0^t \parallel \frac{\mathrm{d}}{\mathrm{d}\, s}\psi_1(\jmath_1(s)) \parallel^n_{x_1} \mathrm{d}\, s \\ &= k\mathrm{b}(x)P_1(x)t. \end{aligned} \tag{3.5}$$

Therefore, in view of the definition of the Finsler metric (2.3) we have

$$\rho(x, \sigma_1(t,x)) \leq k\mathrm{b}(x)P_1(x)t. \tag{3.6}$$

If $x \in \mathbb{V}_1$, by the mean value theorem we can find $\delta \in (0,1)$ such that

$$\begin{aligned} \varphi(\sigma_1(t,x)) - \varphi(x) &= \varphi \circ \psi_1^{-1}\big(\psi_1(x) - t\mathrm{b}(x)P_1(x)h_1\big) - \varphi \circ \psi_1^{-1}(\psi_1(x)) \\ &= \big\langle (\varphi \circ \psi_1^{-1})'\Big(\psi_1(x) - \delta t\mathrm{b}(x)P_1(x)f_1\Big), h_1\big\rangle (t\mathrm{b}(x)P_1(x)) \\ &< -2\epsilon(1+k^2)\mathrm{b}(x)P_1(x)t. \end{aligned} \tag{3.7}$$

For $x \notin \mathbb{V}_1$ we have $P_1(x) = 0$ so (3.7) holds also. Now assume σ_{j-1} is well defined and continuous and (3.6) and (3.7) hold, we prove them for j. By (3.5) and the triangular inequality for all $n \in \mathbb{N}$

$$\rho_n(x, \sigma_{j-1}(t,x)) \leq kt\mathrm{b}(x) \sum_{q=0}^{j-1} P_q(x) \leq kt, \tag{3.8}$$

also,

$$\rho(x, \sigma_{j-1}(t,x)) \leq kt. \tag{3.9}$$

But, $t < \dfrac{r_{\min}}{1+k^2}$ therefore

$$\rho_n(x, \sigma_{j-1}(t,x)) \leq \frac{r_{\min}}{2}. \tag{3.10}$$

For any $x \in \operatorname{supp}(P_j)$ with $\sigma_{j-1}(t,x) \in \mathbb{V}_j$, let C_x be the set of all C^1-curves $\iota : [a,b] \to M$ with $\iota(a) = x$ and $\iota(b) = \sigma_{j-1}(t,x)$ that lie in $\mathbb{V}_j$. If a curve ι that joins x to $\sigma_{j-1}(t,x)$ leaves $\mathbb{V}_j$, then since $\rho_n(x, M \setminus \mathbb{V}_j) \geqq r_{\min}$ we have $L_n(\iota) \geqq r_{\min}$. Whence

$$\rho_n(x, \sigma_{j-1}(t,x)) = \inf \big\{ L_n(\iota) : \iota \in C_x \big\}. \tag{3.11}$$

If a curve $\iota \in C_x$, then for all $n \in \mathbb{N}$

$$\begin{aligned} L_n(\iota) = \int_a^b \parallel \iota'(s) \parallel_{\iota(t)}^n \mathrm{d}\, s &\geqq \frac{1}{k} \int_a^b \parallel \frac{\mathrm{d}}{\mathrm{d}\, s} \psi_j(\iota(s)) \parallel_{x_j}^n \mathrm{d}\, s = \\ &= \frac{1}{k} \parallel \psi_j(\sigma_{j-1}(t,x) - \psi_j(x) \parallel_{x_j}^n . \end{aligned} \tag{3.12}$$

Hence

$$\parallel \psi_j(\sigma_{j-1}(t,x)) - \psi_j(x) \parallel_{x_j}^n \leq k \rho_n(x, \sigma_{j-1}(t,x)) \leq k^2 t. \tag{3.13}$$

Therefore,

$$\parallel \psi_j(\sigma_{j-1}(t,x) - t\mathrm{b}(x) P_j(x) h_j) - \psi_j(x) \parallel_{x_j}^n \leq k^2 t + t < r_{\min}. \tag{3.14}$$

Thus, if $x \in \operatorname{supp}(P_j)$ and $\sigma_{j-1}(t,x) \in \mathbb{V}_j$, then

$$\psi_j(\sigma_{j-1}(t,x) - t\mathrm{b}(x) P_j(x) h_j \in \psi_j(\mathbb{V}_j)$$

which proves that σ_j is well defined and continuous. Now, define a curve

$$\ell_j(s) = \psi_j^{-1}\big(\psi_j(\sigma_{j-1}(t,x) - s\mathrm{b}(x) P_j(x) h_j\big), \quad s \in [0,t].$$

Then,

$$\rho_n(\sigma_{j-1}(t,x), \sigma_j(t,x)) \leq L_n(\ell_j) \leq \int_0^t \parallel \ell_j' \parallel_{\ell_j}^n \leq k\mathrm{b}(x) P_j(x) t. \tag{3.15}$$

Similarly as in the case $j = 1$ by using the mean value theorem between σ_j and σ_{j-1} we can prove $\varphi(\sigma_j(t,x)) - \varphi(\sigma_{j-1}(t,x)) \leq -2\epsilon(1+k^2)\mathrm{b}(x) P_j(x) t$. This concludes the induction.

Now, let $\mathcal{H}(t,x) = \sigma_p(t,x)$, it satisfies all assertions of the lemma and the poof is complete. □

We will need the following facts.

Lemma 3.2. If $\varphi : M \to \mathbb{R}$ satisfies the Palais-Smale condition in $A \subset M$ and has no critical point in A, then there exists $\epsilon > 0$ such that $\Phi_\varphi(x) < -\epsilon$ for all $x \in A$.

Proof. If there is no such ϵ, then there exists a sequence $(m_i) \subset A$ such that $\Phi_\varphi(m_i) \to 0$ and therefore there exists a subsequent of (m_i) converging to a critical point in A which is a contradiction. □

Lemma 3.3. Suppose that $\varphi : M \to \mathbb{R}$ satisfies the Palais-Smale condition on M at a level $c \in \mathbb{R}$ and U is an open neighborhood of $\mathrm{Cr}(\varphi, c)$. Let

$$U_r = \{x \in M \mid x \in B_r(c), \Phi_\varphi(x) \geq -r\},$$

where $r > 0$ and $B_r(c)$ is the closed ball centered at c with radius r with respect to the usual metric on $\mathbb{R}$. Then, there exists $\epsilon > 0$ such that $U_\epsilon \subset U$.

Proof. Suppose there is no such $\varepsilon > 0$, then there exists a sequence $(x_n) \subset M \setminus U$ such that $\lim_{n\to\infty} \varphi(x_n) = c$ and $\lim_{n\to\infty} \Phi_\varphi(x_n) = 0$. Since φ satisfies the $(PS)_c$ condition, there exists a convergent subsequent of (x_n) with the limit p such that $\varphi(p) = c$ and $\varphi'(p) = 0$ which is a contradiction. □

Theorem 3.4. [11, Theorem 1.1] Let M, N be Hausdorff manifolds, where M is a connected infinite dimensional Fréchet manifold, and N satisfies the first countability axiom, and let $\varphi : M \to N$ be a continuous closed non-constant map. Then φ is proper.

Corollary 3.5. Let M be a connected C^1-infinite dimensional Fréchet manifold, $\varphi : M \to \mathbb{R}$ a C^1 closed non-constant function. Suppose φ satisfies the Palais-Smale condition at all levels.

(1) If for $c \in \mathbb{R}$ and $\delta > 0$ we have

$$\varphi^{-1}[c-\delta, c+\delta] \cap \mathrm{Cr}(\varphi) = \emptyset,$$

then there exists $t_1 < t_0$ and $0 < \epsilon < \delta$ such that

$$\mathcal{H}(t_1, \varphi^{c+\epsilon}) \subset \varphi^{c-\epsilon}. \tag{3.16}$$

(2) If φ has finitely many critical points, and for $c \in \mathbb{R}$ if U is an open neighborhood of $\mathrm{Cr}(\varphi, c)$ ($U = \emptyset$ if $\mathrm{Cr}(\varphi, c) = \emptyset$), then there exist $t_1 < t_0$ and $\epsilon > 0$ such that

$$\mathcal{H}(t_1, \varphi^{c+\epsilon} \setminus U) \subset \varphi^{c-\epsilon}. \tag{3.17}$$

Proof. (1) The function φ is proper by Theorem (3.4) therefore

$$A = \varphi^{-1}[c-\delta, c+\delta]$$

is compact. Since φ satisfies the Palais-Smale condition and has no critical point in $\varphi^{-1}[c-\delta, c+\delta]$ it follows in view of Lemma 3.2 that we can find small enough $0 < \epsilon < \delta$ and $k > 1$ such that $\Phi_{\varphi}(x) < -2\epsilon(1+k^2)$ for all $x \in A$. Let $B = \mathrm{Cr}(\varphi)$ (it is easy to see that B is closed), then by Lemma 3.1 there exists $t_0 > 0$ and B-invariant deformation $\mathcal{H}$ for $[0, t_1]$ (if $t_0 > 1$ let $t_1 = \dfrac{1}{1+k^2}$ otherwise let $t_1 = t_0$) such that for $x \in \varphi^{c+\epsilon}$ we have

$$\varphi(\mathcal{H}(t_1, x)) \leq \varphi(x) - 2\epsilon(1+k^2)t_1 \leq c - \epsilon. \tag{3.18}$$

Thus, $\mathcal{H}(t_1, x) \in \varphi^{c-\epsilon}$ and the proof is complete.

(2) By Lemma 3.3 there exist $\epsilon > 0$ and a neighborhood $U_\epsilon \subset U$ of $\mathrm{Cr}(\varphi, c)$, the neighborhood U_ϵ is closed since φ' is continuous. Let

$$A = \{x \in M \setminus U \mid \varphi(x) \in B_\epsilon(c)\},$$

if necessarily shrink ϵ so that on A there is no critical point (it is possible because the set of critical points is discrete), so by Lemma 3.2 we have $\Phi_\varphi(x) < -\epsilon$. In virtue of Theorem 3.4, $\varphi^{-1}\big(B_\epsilon(c)\big)$ is compact and since A is closed in $\varphi^{-1}\big(B_\epsilon(c)\big)$, A is compact either. Now, as $A \cap U_\epsilon = \emptyset$, by the same arguments as in the previous part there exist a U_ϵ-invariant deformation $\mathcal{H}$ for M, $t_1 < t_0$, such that for $x \in \varphi^{c+\epsilon} \setminus U$ we have $\mathcal{H}(t_1, x) \in \varphi^{c-\epsilon}$, this concludes the proof. □

Theorem 3.6. Let M be a connected C^1-infinite dimensional Fréchet manifold and let $\varphi \in C^1(M, \mathbb{R})$ be a non-constant closed function satisfying the (PS) condition at all levels. Suppose that $\mathcal{F}$ is a deformation invariant class of subsets of M and suppose that

$$c = c(\varphi, \mathcal{F}) = \inf_{A \in \mathcal{F}} \sup_{x \in A} \varphi(x) \tag{3.19}$$

is finite, then c is the critical value for φ.

Proof. We prove by contradiction. If c is not a critical value, then $c \in \mathbb{R} \setminus \mathrm{Cr}(\varphi)$. Since $\mathrm{Cr}(\varphi)$ is closed we can find small enough $\epsilon > 0$ such that

$$\varphi^{-1}\big([c-\epsilon, c+\epsilon]\big) \bigcap \mathrm{Cr}(\varphi) = \emptyset. \tag{3.20}$$

By the definition of c we can find $A \in \mathcal{F}$ such that

$$\sup_{x \in A} \varphi(x) \leq c + \varepsilon, \tag{3.21}$$

therefore, $A \subseteq \varphi^{c+\varepsilon}$. Thus, by Corollary 3.5 we can find $t > 0$ such that

$$\mathcal{H}(t, \varphi^{c+\varepsilon}) \subseteq \varphi^{c-\varepsilon}. \tag{3.22}$$

Hence $\mathcal{H}(t, A) \subseteq \varphi^{c-\varepsilon}$ and since $\mathcal{F}$ is invariant deformation we have $\mathcal{H}(t, A) \in \mathcal{F}$. Thus,

$$c \leqslant \sup_{x \in \mathcal{H}(t,A)} \varphi(x) \leqslant \sup_{x \in \varphi^{c-\varepsilon}} \varphi(x) \leqslant c - \varepsilon, \tag{3.23}$$

which is a contradiction. □

Remark 3.7. Let $\mathcal{F} = \{\{x\} \mid x \in M\}$, then $c(\varphi, \mathcal{F}) = \inf_{x \in M} \varphi(x)$. Let $\mathcal{F} = \{M\}$, then $c(\varphi, \mathcal{F}) = \sup_{x \in M} \varphi(x)$.

The Lusternik-Schnirelmann category $\mathrm{Cat}_X A$ of a subset A of a topological space X is the minimal number of closed sets that cover A and each of which is contractible to a point in X. If $\mathrm{Cat}_X A$ is not finite, we write $\mathrm{Cat}_X A = \infty$.

We will need the following basic properties.

Lemma 3.8. [10, Proposition 2.2] Let T be a topological space, $A, B \subset T$. Then

(1) If $A \subset B$, then $\mathrm{Cat}_X A \leq \mathrm{Cat}_M B$.
(2) $\mathrm{Cat}_M (A \cup B) \leq \mathrm{Cat}_M A + \mathrm{Cat}_M B$.
(3) If A is closed and $\mathcal{H} : [0,1] \times T \to T$ is a deformation, then $\mathrm{Cat}_M A \leq \mathrm{Cat}_M (\mathcal{H}(t_0, A))$.
(4) If T is a Finsler manifold, then there exists a neighborhood U of A such that $\mathrm{Cat}_M \overline{U} = \mathrm{Cat}_M A$.
(5) If M is connected and A is closed then $\mathrm{Cat}_M A \leq \dim A + 1$, where dim is the covering dimension.

Let Co(M) be the set of compact subsets of M. Define the sets

$$\mathcal{A}_i = \big\{A \subset M : A \in \mathrm{Co(M)}, \mathrm{Cat}_M A \geq i\}, \tag{3.24}$$

for $i \in \mathbb{N}$. In view of the property (3) of Lemma 3.8 each $\mathcal{A}_i$ is a deformation invariant class of subsets of M. The i-th Lusternik-Schnirelmann minimax value of φ is defined by

$$\mu_i = \inf_{A \in \mathcal{A}_i} \sup_{x \in A} \varphi(x). \tag{3.25}$$

It is easy to see that the sequence of numbers μ_i is increasing.

Proposition 3.9. Let (M, ρ) be a connected, complete C^1-infinite dimensional Finsler Fréchet manifold and let a C^1-function $\varphi : M \to \mathbb{R}$ be non-constant and closed. Let $\mathrm{Cat}_M M = k$, $k \in \mathbb{N} \cup \{\infty\}$. If φ satisfies the Palais-Smale condition for all $\mu_i, i = 1, \cdots k$, then

(1) either each μ_i is a critical value for φ or $\mu_i = \infty$,
(2) if φ is bounded on $\mathrm{Cr}(\varphi)$, then $\mu_i < \infty$ for $i \leq \mathrm{Cat}_M M$,

(3) if $\mu \coloneqq \mu_j = \mu_{j+1} = \cdots = \mu_{j+m} < 0$ for some $m, j \geq 1$, then φ has at least $m+1$ critical points at level c.

Proof. (1) The proof follows from Theorem 3.6.

(2) Suppose $b = \sup_{\mathrm{Cr}(\varphi)} \varphi < \infty$. Then, by virtue of Corollary 3.5 for some $\varepsilon > 0$, M is deformable to $\varphi^{b+\varepsilon}$ and so $\mathrm{Cat}_M M = \mathrm{Cat}_{\varphi^{b+\varepsilon}} \varphi^{b+\varepsilon}$. Thus, $\varphi^{b+\varepsilon} \in \mathcal{A}_i$ therefore $\mu_i \leq b + \varepsilon$.

(3) If there are infinite number of critical points of φ at level c we are done, therefore, assume that $x_1, \cdots, x_n$ are the only critical points. We can assume that there are open contractible neighborhood $x_i \in U_i$ such that $U_j \cap U_j = \emptyset$ if $i \neq j$. Let $U = \bigcup_{l=1}^n U_l$, by Corollary 3.5 there exist $\epsilon > 0$, $t > 0$ and a deformation $\mathcal{H}$ on M such that

$$\mathcal{H}(t, \varphi^{c+\epsilon} \setminus U) \subset \varphi^{c-\epsilon}. \tag{3.26}$$

Thus, $\mathrm{Cat}_M\big(\varphi^{c+\epsilon} \setminus U\big) \leq j - 1$. Now $\mathrm{Cat}_M U \leq n$, therefore

$$\begin{aligned} j + m \leq \mathrm{Cat}_M \varphi^{c+\epsilon} &\leq \mathrm{Cat}_M\big(\varphi^{c+\epsilon} \setminus U\big) + \mathrm{Cat}_M U \\ &\leq j - 1 + n. \end{aligned} \tag{3.27}$$

Thus, $n \geq m + 1$. □

Theorem 3.10. Let (M, ρ) be a connected, complete C^1- Fréchet Finsler manifold and let a C^1-function $\varphi : M \to \mathbb{R}$ be non-constant, closed and bounded below. Let $\mathrm{Cat}_M M = k$, $k \in \mathbb{N} \cup \{\infty\}$. If φ satisfies the Palais-Smale condition for all $\mu_i, i = 1, \cdots k$, then φ has at least k critical points.

Proof. If there are infinite number of critical points of φ at level c we are done, therefore, we assume that φ has finite number of critical points. Each $\mu_i, i = 1, \cdots, k$, is finite and so a critical value, see Proposition 3.9. To prove the theorem, it is enough to show that

$$\#\big(\mathrm{Cr}(\varphi) \cap \varphi^{\mu_i}\big) \geq i, \quad i = 1, \cdots, k. \tag{3.28}$$

We prove by induction. If $i = 1$, since the global minimum is a critical point (see Remark 3.7) so the relation (3.28) is trivial. Suppose (3.28) is true for $i = 1, \cdots n$, we shall prove it for $n + 1$.

If $\mu_n \neq \mu_{n+1}$, then $\mathrm{Cr}(\varphi, \mu_{n+1}) \neq \emptyset$ by Proposition 3.9. Thus, elements of $\mathrm{Cr}(\varphi, \mu_{n+1})$ and $\mathrm{Cr}(\varphi) \cap \varphi^{\mu_n}$ are obviously different, hence, $\mathrm{Cr}(\varphi) \cap \varphi^{\mu_{n+1}}$ contains at least $n + 1$ distinct points. But if $\mu \coloneqq \mu_n = \mu_{n+1}$, then let m be the least positive number such that $\mu_m = \mu_{n+1}$. By Proposition 3.9

$$\mathrm{Cat}_M\big(\mathrm{Cr}(\varphi, \mu)\big) \geq n + 1 - m + 1 = n - m + 2. \tag{3.29}$$

If $m = 1$, we are done. Let $m > 1$, since $m \leq n$ it follows that

$$\mathrm{Cr}(\varphi) \cap \varphi^{\mu_{m-1}} \geq m - 1.$$

Thereby,

$$\begin{aligned} \#\big(\mathrm{Cr}(\varphi) \cap \varphi^{\mu_{n+1}}\big) &\geq \#\big(\mathrm{Cr}(\varphi) \cap \varphi^{\mu_{m-1}}\big) + \#\big(\mathrm{Cr}(\varphi) \cap \varphi^{\mu_{n+1}}\big) \\ &\geq (m-1) + (n-m+2) = n-1. \end{aligned} \tag{3.30}$$

□

Remark 3.11. As mentioned, the Lusternik-Schnirelmann method is used to calculus of variation problems. However, manifolds of mappings are endowed only with Hilbert manifolds (if possible) or it has been used Banach manifolds of maps belonging to Sobolov spaces. To avoid limitations and dependence on a Sobolev level, we may consider the more general context of Fréchet manifolds and smooth maps. It is also worth mentioning that there are spaces of mappings that do not admit neither Banach nor Hilbert manifolds structures, cf., [5]. Thus, Our results would provide a more general setup for variational problems in non-linear setting.

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