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Mathematical Model of Biogenic Elements and Microalgae Interaction in Wastewater

In this study the mathematical model of a water system of the "consumer - resource" type, in which the resource is harmful impurities (biogenic elements), and the consumer is Euglena gracilis was proposed. In this case, the reservoir has a certain amount of impurities and new impurities do not enter the system.

Introduction. Natural systems, including water systems, evolve and change over time, i.e., they can be considered as dynamic systems. An effective approach to the study and forecasting of water bodies is the use of dynamic systems modelling methods. The use of modelling and forecasting methods determines the understanding of everything that is happening and can happen in water systems, atmosphere, soils, flora, the consequences of human intervention in them, pragmatizes assessments and conclusions, and helps to find optimal technical, technological, organizational and environmental solutions. Mathematical models are recognized as effective tools that can help investigate the economic, environmental and ecological impact of alternative measures to control pollution and conserve resources, and thus help make decisions in formulating environmentally and economically efficient management policies [1].

Our task was to build and study a model of a water system of the "consumer - resource" type, in which the resource is harmful impurities, and the consumer is *Euglena gracilis*. In this case, the reservoir has a certain amount of impurities and new impurities do not enter the system.

Let's denote the concentrations of microalgae and impurities at a given time t as $C_m(t)$ and $C_i(t)$ respectively. Let at the initial time $t=0$

$$C_m(0) = C_{m0} > 0, \; C_i(0) = C_{i0} > 0. \qquad (1)$$

The achievement of the effect depends on the perfection of the choice of trophic functions that determine the qualitative properties of the mathematical model of the systems "predator-prey", "consumer-resource", etc. [2]. The problem of adequate choice of trophic functions has been addressed in many papers [3], which consider the theoretical aspects of their choice, mostly trophic functions are chosen based on the results of experimental studies. It should be noted that today most scientists who study these issues do not have a unanimous opinion on how to solve the problem of choosing trophic functions.

Let us formulate the basic requirements for a mathematical model of the interaction of impurities with microalgae. The concentrations of impurities and microalgae, respectively, and the curves describing the dynamics of their changes, must meet certain requirements:

1) function $C_i = C_i(t)$ should be decreasing, taking values from $C_{i0} = C_i(0)$ to $C_i(T)$, here T is the number of days during which the microalgae interact with the impurities. In most cases, the concentration $C_i(T)$ is close to zero, and, as a rule, $T \in [4;7]$;

2) the graph of the function $C_i = C_i(t)$ must have a pronounced *S*-shape. In particular, the function $C_i = C_i(t)$ at the initial moment $t = 0$ of time should decrease at such a rate $C_i'(0)$ that the value $C_i(1)$ is approximately equal to the calculated value $C_i^*(1)$. The function $C_i(t)$ should reach the fastest decay at the point $t = t^*$, and the ordinate of the point $\left(t^*;\, C_i\left(t^*\right)\right)$ – the point of inflection of the curve $C_i = C_i(t)$, must satisfy the inequality $C_i\left(t^*\right) > \frac{C_{i0}}{2}$. In addition to this $t^* < T/2$. The steepest graph of the function $C_i = C_i(t)$ should be mostly in the interval $t \in (1;3)$. The curve $C_i(t)$ should be convex on the interval $(0;\, t^*)$ and concave on the interval $\left(t^*;\, T\right)$;

3) the function $C_m = C_m(t)$ should increase in the interval $t \in [0;T]$, i.e. during all days of interaction of microalgae with impurities, or in the interval $t \in [0;T_1]$, where $T_1 < T$, acquire values from $C_{m0} = C_m(0)$ to $(C_m)_{\max}$. The rate of growth of the function $C_m(t)$ should initially increase from $C_m'(0)$ to $\left(C_m'\right)_{\max} = C_m'\left(t^{**}\right)$ and then monotonically decrease to a certain small value. The graph of the function $C_m = C_m(t)$ should also have an *S*-shape. The curve $C_m(t)$ should be concave on the intercept $(0;\, t^{**})$ and convex on the interval $\left(t^{**};T\right)$. The point $\left(t^{**};C_m\left(t^{**}\right)\right)$ of the curve $C_m(t)$ should be an inflection point, and the following conditions should be met: $t^{**} > T/2$, approximately $C_m\left(t^{**}\right) > \frac{2}{3}(C_m)_{\max}$, i.e., the curve $C_m(t)$ should change from concave to convex essentially at the final stage (in the last one or more days, depending on the initial values of the concentrations of impurities and microalgae) of the interaction of microalgae with impurities.

The above requirements for a mathematical model of the interaction of impurities with microalgae are generally satisfied by the system of differential equations

$$\begin{cases} \dfrac{dC_m}{dt} = \alpha \cdot q \cdot C_m \cdot T_1(C_i, C_m) - \gamma C_m, \\ \dfrac{dC_i}{dt} = -\beta \cdot q \cdot C_m \cdot T_2(C_i, C_m) \end{cases} \quad (2)$$

with initial conditions (1).

The basis for building system (2) is the mathematical model of Lotka and Volterra [4], which they proposed to describe the interaction of two species – the population of predators and the population of victims. In our case, microalgae *Euglena gracilis* act as predators, and impurities in water act as victims.

In system (2), the parameter α characterizes the process of increasing the concentration of microalgae due to the absorption of impurities; parameter β characterizes the rate of reduction of the concentration of impurities in the water environment as a result of interaction with algae; parameter γ (specific mortality) characterizes the rate of decrease in algae concentration due to natural processes.

The value of the parameter α is equal to the sum of the initial rate of growth of the concentration of microalgae and the specific mortality γ multiplied by the initial value of the concentration of microalgae, i.e.

$$\alpha = \frac{dC_m(0)}{dt} + \gamma C_{m0};$$

the value of the parameter β is equal to the absolute value of the rate of decrease in the concentration of impurities at the initial moment of time $t = 0$:

$$\beta = -\frac{dC_i(0)}{dt}.$$

We will assume that the value of the parameter γ does not exceed 0.05.

The values of parameters α and β cannot be precisely determined. The authors proposed formulas for determining the approximate values of these parameters depending on the product of the initial concentration values C_{m0} and C_{i0} :

$$\alpha \approx 8.0 \cdot \ln\left(1 + 0.313 \cdot (C_{i0} C_{m0})\right) \cdot k_m + \gamma C_{m0},$$

$$\beta \approx \left(1 - \ln\left(1 + 0.98 \cdot (C_{i0} C_{m0})^{0.03}\right)\right) \cdot C_{i0} \cdot k_i.$$

The equations of the system (2) contain two trophic functions:

$$T_1(C_i, C_m) = \frac{1}{C_{m0}} \cdot \frac{1 - \left(1 - \dfrac{C_i}{C_{i0}}\right)^n}{\left(1 + \delta_1 \left(\dfrac{C_i}{C_{i0}}\right)^{n_1}\right)\left(1 + \delta_2 \left(\dfrac{C_m}{C_{m0}}\right)^{n_2}\right)} \quad (3)$$

$$T_2(C_i, C_m) = \frac{1}{C_{m0}} \cdot \frac{\dfrac{C_i}{C_{i0}}}{\left(1 + \delta_3 \left(\dfrac{C_i}{C_{i0}}\right)^{n_3}\right)\left(1 + \delta_4 \left(\dfrac{C_m}{C_{m0}}\right)^{n_4}\right)}$$

From the (3) $R(C_i)=1-\left(1-\frac{C_i}{C_{i0}}\right)^n$, $(n=30)$ is designed primarily to prevent a rapid increase in the concentration of microalgae $C_m(t)$ during the first few days of their interaction with impurities.

In order to reproduce the dynamics, the mathematical model includes regulating factors that appear in the denominators of trophic functions. In particular, based on the results of the research, the following values of the indicators are proposed n_1-n_4: $n_1=4.5, n_2=0.89, n_3=5.0, n_4=2.2$.

To determine the values of the coefficients $\delta_1-\delta_4$, the parameter q, which has a certain relationship with these coefficients, is used. Let us illustrate it. To do this, consider the system (2) at the initial time $t=0$:

$$\begin{cases} \dfrac{dC_m(0)}{dt}=\alpha\cdot q\cdot\dfrac{1}{(1+\delta_1)(1+\delta_2)}-\gamma C_{m0}, \\ \dfrac{dC_i(0)}{dt}=-\beta\cdot q\cdot\dfrac{1}{(1+\delta_3)(1+\delta_4)}. \end{cases}$$

In order to preserve the meaning of the parameters α and β we require that the following inequalities hold at the initial moment $t=0$:

$$\begin{cases} \alpha=\alpha\cdot q\cdot\dfrac{1}{(1+\delta_1)(1+\delta_2)}, \\ -\beta=-\beta\cdot q\cdot\dfrac{1}{(1+\delta_3)(1+\delta_4)}. \end{cases}$$

which means that $(1+\delta_1)(1+\delta_2)=q$, $(1+\delta_3)(1+\delta_4)=q$. Given that the coefficients $\delta_1-\delta_4$ take on positive values, let us assume, for example, that $q=4$, then $\delta_1\in(0;3)$ is a certain constant, respectively $\delta_2=q/(1+\delta_1)-1$. Similarly, $\delta_3\in(0;3)$, $\delta_4=q/(1+\delta_3)-1$. Based on the results of our research, we propose calculation formulas for determining the values of the coefficients δ_1 and δ_3:

$$\delta_1\approx\ln\left(1+4.928\cdot(C_{i0}C_{m0})^{0.058}\right);\quad \delta_3\approx\ln\left(1+19.96\cdot(C_{i0}C_{m0})^{-0.037}\right).$$

For the case of $C_{m0}=120$, $C_{i0}=14$ coefficient values $\delta_1-\delta_4$ to the nearest thousand are as follows: $\delta_1=2.15, \delta_2=0.27, \delta_3=2.78, \delta_4=0.058$.

Example. Table 1 and figures 1–2 show the results of calculations of changes in concentrations of microalgae and phosphorus impurities for 7 days of their interaction in wastewater.

Table 1.

Calculation results

Days	0	1	2	3	4	5	6	7
C_m	120.0	170.1	308.9	566.3	787.3	830.4	821.8	806.7
C_i	14.0	11.202	5.532	1.342	0.195	0.023	0.003	0.000

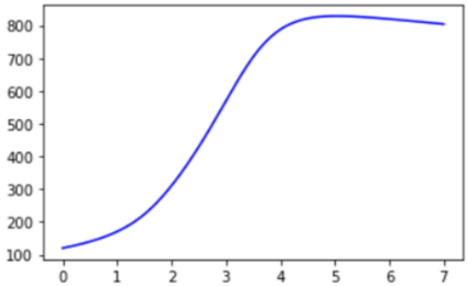

Fig 1. Dynamics of microalgae biomass concentration changes in wastewater

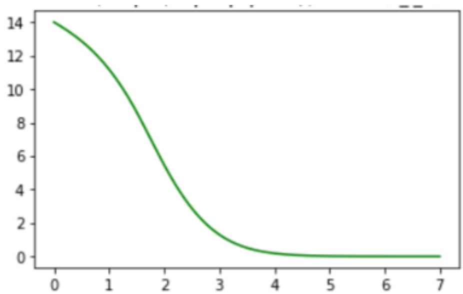

Fig 2. Dynamics of changes in phosphorus concentration in wastewater

Conclusion**.** The proposed system of differential equations (2) with initial conditions (1) generally describes the dynamics of changes in the concentrations of *Euglena* and phosphorus impurities in the course of their interaction. The developed model allows for a deeper study of phytoremediation of wastewater by microalgae.

The proposed model can be considered as a basis for creating a mathematical model of the interaction of phosphorus impurities and types of nitrogen impurities with microalgae Euglena gracilis.

References

1. http://repositsc.nuczu.edu.ua/bitstream
2. Lesia Pavliukh, Sergii Shamanskyi, Roman Odarchenko, Oleh Zhelezniak, Andrii Tereshchenko. Information and mathematical model of wastewater treatment processes. Workshop on Information Technology and Mathematical Modeling for Environmental Safety. *CEUR Workshop Proceedings*. 2021. Vol. 3021. P. 189-196.
3. Arditi R., Ginzburg L.R., 2012. How Species Interact: Altering the Standard View on Trophic Ecology. N.Y.: Oxford Univ. Press. 170 p.
4. Volterra V., 1931. Leçons sur la théorie mathématique de la lutte pour la vie. P.: Gauthier–Villars. 214 p.